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Brown Representability in Equivariant Motivic Homotopy Theory

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Introduction

Cohomology theories, such as singular cohomology or topological K-theory, give important tools for studying topological[1] spaces. One of the properties of these theories is the fact that they are "stable" with respect to the suspension functor. For example, if X is a based topological space, there is a natural isomorphism $\tilde{H}^n(X) \xrightarrow{\cong} \tilde{H}^{n+1}(\Sigma X)$ of reduced singular cohomology groups. Indeed, using the classification of singular homology groups as homotopy classes $[X, K(\mathbb{Z}, n)]$ of maps from X to the n-th Eilenberg-Maclane space, this isomorphism can be given by sending a map $f : X \to K(\mathbb{Z}, n)$, to the adjoint of the map

$$X \xrightarrow{f} K(\mathbb{Z}, n) \xrightarrow{\simeq} \Omega K(\mathbb{Z}, n+1).$$

This suggests a way of creating new cohomology theories. Suppose we are given topological spaces $\{T_n\}_{n\geq 0}$ and weak equivalences $T_n \xrightarrow{\simeq} \Omega T_{n+1}$ (this defines what we call an Ω-*spectrum*). Define $E^q(X) = [X, T_q]$. Then E^q satisfies the same stability conditions as the singular cohomology groups, as the suspension/loop space adjunction gives us isomorphisms

$$[X, T_q] \xrightarrow{\cong} [X, \Omega T_{q+1}] \xrightarrow{\cong} [\Sigma X, T_{q+1}].$$

In fact, it can be shown that the E^qs give a cohomology theory (cf. Section 22.2 in [May99]). Brown representability states that every cohomology theory can be represented by an Ω-spectrum in this way.

Taking a more categorical perspective, there exists a suitably defined category of spectra with an associated stable homotopy theory, with the property that every cohomology theory is represented by an Ω-spectra. Experience has shown that in some cases it is more fruitful to study the spectrum representing a cohomology theory than the properties of the individual cohomology groups.

Motivic homotopy theory was introduced by Voevodsky [Voe98]. It provides a framework for applying techniques from algebraic topology to the study of smooth schemes over a base scheme S, with the intuition that the affine line should be contractible. In analogy to algebraic topology, there is both an unstable homotopy category, $\mathrm{H}(S)$, and a category of motivic spectra with an associated stable homotopy category, denoted as $\mathrm{SH}(S)$. The objects of $\mathrm{SH}(S)$ define cohomology theories on **Sm/S**. Every cohomological functor on $\mathrm{SH}(S)$ is representable, but this is a rather weak statement. Cohomology theories on **Sm/S**, such as sheaf or étale cohomology, do not necessarily extend to all of $\mathrm{SH}(S)$. A more promising result was announced by Voevodsky and proven by Naumann and Spitzweck [NS11], under the assumption that the category **Sm/S** is countable. This states that every cohomology theory on the subcategory of compact objects is representable by a compact object. In [NSØ09], the authors use this result together with a motivic Landweber exact functor theorem to produce motivic (ring) spectra.

Throughout mathematics, group actions on the objects of study make for interesting phenomena. There has been done much work on equivariant homotopy theory ([Lew+86] is one reference), and with the advent of motivic homotopy theory, we have a framework for applying these techniques to the study of schemes equipped with a group scheme action. The constructions of the equivariant theory follows the same formal pattern as the non-equivariant. Consequently, one would expect many results - particularly those concerning the formal parts of the theory - to carry over to the equivariant setting. The purpose of this thesis is to document a proof of Brown representability for equivariant motivic homotopy theory.

[1] In practice, it is necessary to impose some technical restrictions on the topological spaces we consider.

We assume familiarity with the language of schemes, such as can be found in [Har77; Vak]. For category theory, we refer to [Mac98] or the first chapter in [Vak]. We will also use some basic facts about simplicial sets, for which two standard references are [May67; GJ09].

Outline

The thesis is organized as follows:

Chapter 1 We introduce model categories, which is an important tool in the formulation of the theory. We discuss the small object argument, cofibrantly generated model categories and simplical model categories and Bousfield localization. The last section describes how to construct spectra in left proper, cellular model categories.

Chapter 2 We start by introducing Grothendieck topologies and the Nisnevich topology on the category of smooth G-schemes for a group G. We use this and the tools from Chapter 1 to construct the equivariant motivic model category and the model category of equivariant motivic spectra.

Chapter 3 We introduce triangulated categories, and discuss how the stable homotopy category is triangulated. We then discuss Brown representability in the context of equivariant motivic homotopy theory.

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Contents

Chapter 1

Model categories

Algebraic topology studies topological spaces up to homotopy equivalence. There are many tools developed for this purpose, notably homotopy and singular (co)homology groups all of which are homotopy invariants, so that they do not distinguish between homotopy equivalent spaces. In fact, both homotopy groups and singular (co)homology groups do not distinguish between weakly equivalent spaces

A characteristic feature of classical algebraic topology is that many results can be proved using the fact that certain maps can be lifted. An example of this is the homotopy lifting and extension property [May99], which states that if (X, A) is a relative CW-complex, and $e : Y \to Z$ is a weak equivalence, then the indicated lifts exists in the diagram where the solid maps commute

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, hook] & & A \times I \arrow[dl, "h"] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, hook] \\
& Y & & Z \arrow[ll, "e"] & \\
X \arrow[rr, "i_0"'] \arrow[ur] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

such that the entire diagram commutes. An immediate consequence is Whitehead's theorem, that a map between CW-complexes is a weak equivalence if and only if it is a homotopy equivalence.

The notions of homotopy and weak equivalence also arise in homological algebra. If R is a commutative ring, then a *quasi-isomorphism* of chain complexes of R-modules is a map of chain complexes which induces isomorphisms on homology groups.

Model categories were introduced by Quillen [Qui67] and provide an abstract framework for doing "homotopy theory". In both the case of topology and chain complexes, one has a category, say $\mathcal{C}$, and a class $\mathcal{W}$ of maps in $\mathcal{C}$ which we want to invert in order to obtain the "homotopy category" of $\mathcal{C}$.

We will use Hovey [Hov99] and Hirschhorn [Hir03] as our main sources in the sections on model categories and cofibrantly generated model categories. Another recent source to model categories is [MP12]. In the section on Bousfield localization, we follow Hirschorn, which is the standard reference on the topic. The section on spectra in model categories use the theory from Hovey [Hov01].

1.1 Model categories

Suppose $f : X' \to Y'$ and $g : X \to Y$ are maps in an arbitrary category which fits into a commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Y' \arrow[r] & Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps compose to the identity. Then we say that f is a *retract* of g.

Suppose we have a diagram such that the solid maps commute

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r] \arrow[ur, dashed, "h"] & W \end{tikzcd}$$

and there exists a lift h making the whole diagram commute. Then we say that f has the *left lifting property with respect to* g, and g has the *right lifting property with respect to* f.

Let $\mathcal{C}$ be a category and $\mathcal{D}$ a class of maps in $\mathcal{C}$. Then $\mathcal{D}$ has the 2-*out-of*-3-*property* if whenever f and g are maps in $\mathcal{C}$ such that $f \circ g$ is defined, and two of f, g and $f \circ g$ are in $\mathcal{D}$, then so is the third.

Definition 1.1.1. A category $\mathcal{M}$ is a *model category* if there are three classes of maps $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ in $\mathcal{M}$ such that the following five axioms are satisfied.

MC1 $\mathcal{M}$ is complete and cocomplete.

MC2 The class $\mathcal{W}$ has the 2-out-of-3 property.

MC3 The classes $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts, that is, if g is in either of the three classes and f is a retract of g, then f is in the same class.

MC4 The maps in $\mathcal{C}$ have the left lifting property with respect to maps that are both in $\mathcal{F}$ and $\mathcal{W}$. The maps in $\mathcal{F}$ have the right lifting property with respect to maps that are both in $\mathcal{C}$ and $\mathcal{W}$.

MC5 There are functorial factorizations (α, β) and (γ, δ) of maps in $\mathcal{M}$ such that for any map f in $\mathcal{M}$, $\alpha(f)$ is a cofibration, $\beta(f)$ is both a fibration and a weak equivalence, $\gamma(f)$ is a fibration and $\delta(f)$ is both a cofibration and a weak equivalence.

The maps in $\mathcal{F}$ are called *fibrations*, the maps in $\mathcal{C}$ *cofibrations* and the maps in $\mathcal{W}$ *weak equivalences*. The maps that are both in $\mathcal{W}$ and $\mathcal{F}$ (respectively $\mathcal{C}$) are called *trivial fibrations* (respectively *trivial cofibrations*).

If $\mathcal{M}$ is a model category, then, by taking the colimit and limit of the empty diagram, (MC1) implies that $\mathcal{M}$ has an initial and a terminal object. We define an object to be *cofibrant* if the unique map from the intial object is a cofibration. Similarily, an object is *fibrant* if the map to the final object is a fibration.

If X is an object in a model category, then by applying the factorization of maps into cofibrations followed by trivial fibrations to the map $\emptyset \to X$ from the intial object to X, we get a functorial assignment to a cofibrant object QX which is weakly equivalent to X. Any such object is called a *cofibrant approximation* of X. In the same manner, X has a *fibrant approximation*, say RX.

Remark. In the above definition, a model category consists of a complete and cocomplete category $\mathcal{M}$, and three classes of maps, $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$, in $\mathcal{M}$ satisfying certain properties. Another point of view is to say that the three classes of maps defines a *model strucure* on $\mathcal{M}$. This emphasizes the fact that there can be different model structures on the same underlying category $\mathcal{M}$.

Proposition 1.1.2. *Let $\mathcal{M}$ be a model category. A map in $\mathcal{M}$ is a cofibration (respectively a fibration) if and only if it has the left (respectively right) lifting property with respect to all trivial fibrations (respectively trivial cofibrations).*

Proof. It suffices to prove the statement concerning cofibrations.

By (MC4), it suffices to show that a map $f : X \to Y$ with the left lifting property with respect to trivial fibrations is a cofibration. By (MC4), there is a factorization $f = pi$ of f into a cofibration followed by a trivial fibration. Hence, there is a lift q in the diagram below.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}_y"'] \arrow[ur, dashed, "q"] & Y \end{tikzcd}$$

Since $pq = \mathrm{id}_Y$, it follows that f is a retract of i, as can be seen in the diagram below.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[r, "\mathrm{id}_X"] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "q"] & Z \arrow[r, "p"] & Y \end{tikzcd}$$

It follows that f is a cofibration by (MC3). $\square$

Example 1.1.3. Let R be a ring. The category $\mathbf{Ch}_+(R)$ of chain complexes of R-modules concentrated in positive degreee has a model stucture in which the quasi-isomorphisms, i.e. the maps which induce isomorphisms in homology, are the weak equivalences. The fibrations are the degreewise surjections, and the cofibrations are the degreewise injections with projective cokernel.

Example 1.1.4 (Cf. [GJ09])**.** Let $f : X \to Y$ be a map of simplicial sets. Define f to be a weak equivalence if its geometric realization is a weak equivalence of topological spaces, i.e. $|f|$ induces isomorphisms for all homotopy groups. Furthermore, define f to be a Kan fibration if it has the right lifting property with respect to every map $\Lambda[n,k] \to \Delta[n]$ (i.e. inclusions of horns) for all $n > 0$ and $0 \leq k \leq n$, and a cofibration if it has the left lifting property with respect to maps that are both weak equivalences and Kan fibration. This gives **SSet** the structure of a model category.

It can be showed that the cofibrations are the monomorphism, that is, simplicial set maps which are injective in each degree. Hence, every object in **SSet** is cofibrant. The fibrant objects are by definition the Kan complexes.

Example 1.1.5. The model structure on **SSet** can be extended to $G\mathbf{SSet}_\bullet$, the category of pointed simplicial sets with a G-action for some finite group G. An object of $G\mathbf{SSet}_\bullet$ is an object X in $\mathbf{SSet}_\bullet$ together with a group action $a_X : G_+ \wedge X \to X$ compatible with the group structure on G, while maps in $G\mathbf{SSet}_\bullet$ are maps in $\mathbf{SSet}_\bullet$ compatible with group actions. If H is a subgroup of G, then let $\mathrm{Fix}(H, X)$ be the subspace of X which is left unchanged under the group action by elements of H. If $f : X \to Y$ is a map of pointed G-simplicial sets, then the underlying map of simplicial sets must restrict to a map $\mathrm{Fix}(H, X) \to \mathrm{Fix}(H, Y)$ by equivariance.

Hence, for any subgroup H, $\mathrm{Fix}(H, -)$ is a functor $G\mathbf{SSet}_\bullet \to \mathbf{SSet}_\bullet$. A map f of $G\mathbf{SSet}_\bullet$ is then defined to be a weak equivalence if $\mathrm{Fix}(H, f)$ is a weak equivalence (of pointed simplicial sets) for all subgroups H of G, and similarly a fibration if $\mathrm{Fix}(H, f)$ is. The cofibrations are, necessarily, the maps with the left lifting property with respect to maps that are both weak equivalences and fibrations.

Example 1.1.6. Let $\mathcal{C}$ be a category. A *simplicial presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to \mathbf{SSet}$. The class of simplicial presheaves on $\mathcal{C}$ and natural transformations between them form a category, which we denote as $sPre(\mathcal{C})$. There are several model structures on $sPre(\mathcal{C})$. If $f : X \to Y$ is a map of simplicial presheaves, then f is a

- *objectwise weak equivalence* if $f(U) : X(U) \to Y(U)$ is a weak equivalence of simplicial sets for every object U in $\mathcal{C}$,
- *injective cofibration* if $f(U) : X(U) \to Y(U)$ is a cofibration of simplicial sets for every object U in $\mathcal{C}$,
- *projective fibration* if $f(U) : X(U) \to Y(U)$ is a fibration of simplicial sets for every object U in $\mathcal{C}$.

The injective (respectively projective) model structure on $sPre(\mathcal{C})$ is given by defining an injective fibration (respectively projective cofibration) to be a map which has the right (respectively left) lifting property with respect to maps that are both objectwise weak equivalences and injective cofibrations (respectively projective fibration).

Example 1.1.7. If $\mathcal{M}$ is a category, then a *pointed object* in $\mathcal{M}$ is a map $* \to X$ from the terminal object to an object X. By abuse of notation, we denote a pointed object $* \to X$ by X. A map of pointed objects $f : X \to Y$ is a commutative diagram

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "\tilde{f}"] & & Y \end{tikzcd}$$

where $\tilde{f}$ is a map in M. The classes of pointed objects and pointed maps in $\mathcal{M}$ form a category, which we say is a *pointed category* and denote by $\mathcal{M}_\bullet$. If $\mathcal{M}$ is a model category, then we can give $\mathcal{M}_\bullet$ a model structure by letting a map of pointed objects be a weak equivalence, cofibration or fibration if the underlying map in $\mathcal{M}$ is.

Remark. The example of a pointed category is a special case of an under-category. If $\mathcal{C}$ is a category and A an object of $\mathcal{C}$, then the *category of objects under* A, denoted $(A \downarrow \mathcal{C})$, is the category whose objects are maps $A \to X$ in $\mathcal{C}$, and maps (i.e., maps in $(A \downarrow \mathcal{C})$) are commutative diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

If $\mathcal{C}$ is a model category, then $(A \downarrow \mathcal{C})$ can be given a model structure in a similar way as the pointed case. The same is true for the dual notion of the category of objects over A.

A model category is said to be *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence. Dually, a model category is *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence.

Proposition 1.1.8 ([Hir03, Prop.13.1.2])**.** *Every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence.*

Corollary 1.1.9. *If $\mathcal{M}$ is a model category where every object is cofibrant, then $\mathcal{M}$ is left proper.*

Example 1.1.10. The class of cofibrations of simplicial sets is the class of inclusions, hence every simplicial set is cofibrant, and **SSet** is left proper. This immediatly generalizes to the injective model structure on $sPre(\mathcal{C})$ for any category $\mathcal{C}$, since the initial object of $sPre(\mathcal{C})$ is the constant presheaf with value the empty simplicial set for every object in $\mathcal{C}$.

1.1.1 The homotopy category of a model category

One of the most basic facts about a model category $\mathcal{M}$ is that it has an associated homotopy category.

Definition 1.1.11. Let $\mathcal{C}$ be a category, and $\mathcal{W}$ a class of maps in $\mathcal{M}$. Then the *localization of $\mathcal{C}$ with respect to $\mathcal{W}$* is, if it exists, a category $L_{\mathcal{W}}\mathcal{C}$ and a functor $\gamma : \mathcal{C} \to L_{\mathcal{W}}\mathcal{C}$ such that

1. if w is a map in $\mathcal{W}$, then $\gamma(w)$ is an isomorphism.
2. if $\mathcal{D}$ is a category and $\varphi : \mathcal{C} \to \mathcal{D}$ is a functor sending maps in $\mathcal{W}$ to isomorphisms, then there is a unique functor $\delta : L_{\mathcal{W}}\mathcal{C} \to \mathcal{D}$ such that $\varphi = \delta\gamma$, i.e., the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & \mathcal{D} \\ & L_{\mathcal{W}}\mathcal{C} \arrow[ur, dashed, "\exists!\delta"'] & \end{tikzcd}$$

commutes.

Relaxing for a moment the definition of a category, the localization of a category $\mathcal{C}$ with respect to $\mathcal{W}$ can be formed by formally adding inverses to the maps in $\mathcal{W}$. However, there is no reason for this to be a locally small category, meaning that the class of map between two objects might be a proper class. If we stick to the convention that a category is required to be locally small, then the localization of $\mathcal{C}$ with respect to an arbitrary class of maps need not exist unless $\mathcal{C}$ is small. However, in the context of model categories, there is the following theorem due to Quillen.

Theorem 1.1.12 ([Hov99, Prop.13.1.2])**.** *If $\mathcal{M}$ is a model category and $\mathcal{W}$ its class of weak equivalences, then the localization of $\mathcal{M}$ with respect to $\mathcal{W}$ exists.*

The localization of $\mathcal{M}$ with respect to $\mathcal{W}$ is called the *homotopy category of $\mathcal{M}$*, and is denoted by $\operatorname{Ho}\mathcal{M}$.

As we have seen, the homotopy category is somewhat elusive. However, there is a standard construction on $\mathcal{M}_{cf}$, the subcategory of objects in $\mathcal{C}$ that are both cofibrant and fibrant, which gives a more concrete category which is equivalent to $\operatorname{Ho}\mathcal{M}$. We will not describe how this is done, but the key point is that if X is cofibrant and Y is fibrant, there is an equivalence relation $\sim$ on the mapping set $\mathcal{M}(X,Y)$. This equivalence relation and the inclusion functors induce equivalences of categories

$$\mathcal{M}_{cf}/\!\sim \xrightarrow{\cong} \operatorname{Ho}\mathcal{M}_{cf} \xrightarrow{\cong} \operatorname{Ho}\mathcal{M}_{c} \xrightarrow{\cong} \operatorname{Ho}\mathcal{M}.$$

The inverse functors $\operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{M}_c$ and $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}_{cf}$ are induced by the cofibrant and fibrant replacement functors of $\mathcal{M}$.

1.1.2 Quillen functors

Having defined model categories, it is natural to look at functors between model categories. If $\mathcal{M}$ and $\mathcal{N}$ are model categories, then a *left Quillen functor* is a left adjoint functor $F : \mathcal{M} \to \mathcal{N}$ which preserves cofibrations and trivial cofibrations. A *right Quillen functor* is a right adjoint functor $U : \mathcal{M} \to \mathcal{N}$ which preserves fibrations and trivial fibrations. If $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is an adjoint functor pair, with F a left Quillen functor and G a right Quillen functor, then (F, G) is a *Quillen pair.*

Proposition 1.1.13 ([Hir03, Prop. 8.5.3])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, and $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ an adjoint functor pair. Then the following is equivalent.*

1. *F is a left Quillen functor.*
2. *G is a right Quillen functor.*
3. *(F, G) is a Quillen pair.*

Proof. By adjointness, a lift in the diagram

$$\begin{tikzcd} FX \arrow[r] \arrow[d, "F(i)"'] & Z \arrow[d, "p"] \\ FY \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & GZ \arrow[d, "G(p)"] \\ Y \arrow[r] \arrow[ur, dashed] & GW \end{tikzcd}$$

is equivalent. By Prop. 1.1.2, p is a trivial fibration (respectively fibration) if and only if $G(i)$ is a cofibration (respectively trivial cofibration). Thus, G preserves cofibrations (respectively trivial cofibrations) if and only if F preserves trivial fibrations (respectively fibrations). $\square$

If $F : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor, then its *total left derived functor* is the composite

$$\mathbb{L}F : \operatorname{Ho}\mathcal{M} \xrightarrow{\operatorname{Ho}Q} \operatorname{Ho}\mathcal{M}_c \xrightarrow{\operatorname{Ho}F} \operatorname{Ho}\mathcal{N},$$

where Q is the functorial cofibrant replacement functor for $\mathcal{M}$. The *total right derived functor* *RU* of a right Quillen functor U is defined similarily, using the fibrant replacement functor. This qives an adjunction of homotopy categories called the *total derived adjunction.*

A Quillen pair $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is said to be a *Quillen equivalence* if for every cofibrant object A in $\mathcal{M}$ and fibrant object X in $\mathcal{N}$, a map $f : A \to GX$ is a weak equivalence in $\mathcal{M}$ if and only if its adjoint map $f^{\#} : FA \to X$ is a weak equivalence in $\mathcal{N}$. A Quillen pair is a Quillen equivalence if and only if its total derived adjunction is an equivalence of categories [Hir03, Thm. 8.5.23].

Example 1.1.14. The adjoint pair of the singular set functor and geometric realization between **Top** and **SSet** is an important tool in classical algebraic topology. Following the notation of [GJ09], recall that Sing : **Top** $\to$ **SSet** is given by sending a topological space X to the simplicial set

$$\mathbf{n} \mapsto \mathbf{Top}(|\Delta^n|, X),$$

where $|\Delta^n|$ is the standard topological n-simplex.

Roughly speaking, the geometric realization of a simplicial set K is given by gluing edges along vertices. A concise definition can be given by considering the category $\Delta \downarrow K$ of maps $\sigma : \Delta^n \to X$ and diagrams

$$\begin{tikzcd} \Delta^n \arrow[dr, "\sigma"] \arrow[dd, "\theta"'] & \\ & X \\ \Delta^m \arrow[ur, "\tau"] & \end{tikzcd}$$

Geometric realization is then the functor $|-| : \mathbf{SSet} \to \mathbf{Top}$ given by sending a simplicial set K to

$$|K| = \operatorname{colim}_{\sigma \in \Delta\downarrow K} |\Delta^n|$$

with the colimit topology.

The adjointness of Sing and $|-|$ follows from the observation that there is an isomorphism of simplicial sets

$$K \cong \operatorname{colim}_{\sigma \in \Delta\downarrow K} \Delta^n.$$

Hence, there are isomorphisms

$$\begin{aligned} \mathbf{Top}(|K|, X) &\cong \mathbf{Top}(\operatorname{colim}_{\sigma \in \Delta\downarrow K} |\Delta^n|, X) \\ &\cong \lim_{\sigma \in \Delta\downarrow K} \mathbf{Top}(|\Delta^n|, X) \\ &\cong \lim_{\sigma \in \Delta\downarrow K} \mathbf{SSet}(\Delta^n, \operatorname{Sing}(X))) \\ &\cong \mathbf{SSet}(K, \operatorname{Sing}(X)). \end{aligned}$$

The proof that $(\operatorname{Sing}, |-|)$ is a Quillen pair, and in fact a Quillen equivalence, is due to Quillen [Qui67]. In Hovey's exposition [Hov01], the results that lead to this is an important part of the proof of the theorem asserting that **SSet** is model category.

Example 1.1.15. Let $\mathcal{C}$ be a category. The projective and injective model structures on $sPre(\mathcal{C})$ have the same weak equivalences, and from the definitions it follows that a projective cofibration is also an injective cofibration. Conversely, an injective fibration is a projective fibration. Hence, the identity functor is a left Quillen functor from the injective to the projective model structure. In fact, it is a Quillen equivalence, so from a homotopy theoretic point of view the two model structures are equivalent.

1.2 Cofibrantly generated model categories

1.2.1 Small objects and relative cell complexes

Let γ be an ordinal and $\mathcal{C}$ a category with small colimits. A *γ-sequence* in $\mathcal{C}$ is a functor $X : \gamma \to \mathcal{C}$ which preserves colimits. In other words, a γ-sequence in $\mathcal{C}$ is a diagram

$$X_0 \to X_1 \to \cdots \to X_\alpha \to \cdots$$

where $\alpha < \gamma$. Since X preserves colimits, it follows that if $\beta < \gamma$ is a limit ordinal, then the natural map $\operatorname{colim}_{\alpha<\beta} X_\alpha \to X_\beta$ is an isomorphism. Moreover, there is a natural map $X_0 \to \operatorname{colim}_{\alpha<\gamma} X_\alpha$. This map is called the *composition* of the γ-sequence.

Let κ be be a cardinal, $\mathcal{C}$ a category with small colimits and $\mathcal{D}$ a subcategory of $\mathcal{C}$. An object X in $\mathcal{C}$ is said to be κ-small relative to $\mathcal{D}$ if for all regular cardinals $\lambda \geq \kappa$ and λ-sequence X_α in $\mathcal{C}$ such that whenever $\alpha + 1 < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ is in $\mathcal{D}$, then the map

$$\operatorname{colim}_{\alpha<\lambda} \mathcal{C}(X, X_\alpha) \to \mathcal{C}(X, \operatorname{colim}_{\alpha<\lambda} X_\alpha)$$

is an isomorphism. If X is κ-small relative to $\mathcal{D}$ for some cardinal κ, it is said to be *small relative to* $\mathcal{D}$, and if X is relative to $\mathcal{C}$ itself, we say that X is *small.*

If $\mathcal{C}$ is category closed under colimits, and I is a set of maps in $\mathcal{C}$, then a map $f : X \to Y$ in $\mathcal{C}$ is a *relative I-cell complex* if it is a transfinite composition of pushouts of maps in I. In other words, there is a cardinal λ and a λ-sequence X_α such that f is the composition of the λ-sequence, and for every ordinal α with $\alpha + 1 < \lambda$, the map $X_\alpha \to X_{\alpha+1}$ is obtained from a pushout along a map in I.

If the map from the initial object to an object X is a relative I-cell complex, then X is said to be an *I-cell complex.* We say that an object X in $\mathcal{C}$ is *small relative to I* if it is small relative to the subcategory of relative I-cell complexes.

1.2.2 The small object argument

If $\mathcal{M}$ is a model category and I is a set of maps in $\mathcal{M}$, then I *permits the small object argument* if every domain in I is small relative to I.

Definition 1.2.1. Let I be a set of maps in a category C. Define a map in $\mathcal{C}$ to be an

1. *I-injective* if it has the left lifting property with respect to the maps in I.
2. *I-cofibration* if it has the left lifting property with respect to the I-injectives.
3. *I-projective* if it has the right lifting property with respect to the maps in I.
4. *I-fibration* if it has the right lifting property with respect to the I-projectives.

Theorem 1.2.2 (The small object argument)**.** *Let $\mathcal{M}$ be a model category and I a set of maps in $\mathcal{M}$ which permits the small object argument. Then every map in $\mathcal{M}$ can be functorially factored into a relative I-cell complex followed by an I-injective.*

For our work on Brown representability, we will need the following version of the small object argument, which appear as Prop. 5 in [NS11]. The proof will also give an outline of the proof of the general version of the small object argument.

Proposition 1.2.3. *Let $\mathcal{C}$ be a category, and I a set of maps in $sPre(\mathcal{C})_\bullet$ such that:*

1. *I is countable,*
2. *I admits the small object argument,*
3. *for domain F of a map in I and sectionwise countable pointed presheaf $G \in sPre(\mathcal{C})_\bullet$, the set $sPre(\mathcal{C})_\bullet(F, G)$ is countable,*
4. *for every $U \in \mathcal{C}$ and codomain G of a map in I, the set $sPre(\mathcal{C})_\bullet(U_+, G)$ is countable.*

Then if $F \in sPre(\mathcal{C})_\bullet$ is sectionwise countable, then the map $F \to *$ can be functorially factored as $F \xrightarrow{i} F' \xrightarrow{p} *$, where i is a relative I-cell complex, p has the right lifting property relative to I, and F' is sectionwise countable.

Proof. By the small object argument, any map $F \to G$ between simplicial presheafs can be factored into an I-cell complex followed by a map with the right lifting property with respect to I. To see that F' is sectionwise countable whenever F is, we repeat the construction of F' in the proof of the small object argument.

Let $F_0 = F$ and assume by induction we have constructed a sequence

$$F = F_0 \to F_1 \to \cdots \to F_n,$$

where each F_i is sectionwise countable. Let $\mathcal{D}$ be the set of commutative squares

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & F_n \arrow[d] \\ Y \arrow[r] & \bullet \end{tikzcd}$$

with $f \in I$. Define F_{n+1} to be the pushout

$$\begin{tikzcd} \coprod_{\mathcal{D}} X \arrow[r] \arrow[d, "\amalg_{\mathcal{D}} f"] & F_n \arrow[d] \\ \coprod_{\mathcal{D}} Y \arrow[r] & F_{n+1} \end{tikzcd}$$

By the Yoneda lemma and assumption 4., it follows that $\coprod_{\mathcal{D}} Y$ is a sectionwise countable presheaf. Hence, F_{n+1} is the pushout of sectionwise countable presheaves, so it is sectionwise countable. Finally, let $F' = \operatorname{colim}_n F_n$. Since the colimit is taken over countably many sectionwise countable presheaves, it follows that F' itself is sectionwise countable. $\square$

1.2.3 Cofibrantly generated model categories

From Prop. 1.1.2, it follows that if $\mathcal{M}$ is a model category, then the fibrations are completely determined by the classes of weak equivalences and cofibrations. Similarily, the cofibrations are determined by the weak equivalences and the fibrations. There are other ways of specifying a model structure on a model category.

Definition 1.2.4. Let $\mathcal{M}$ be a model category and I and J be sets of maps in $\mathcal{M}$ which permits the small object argument. Then $\mathcal{M}$ is *cofibrantly generated* if the following criteria are satisfied:

1. A map is a trivial fibration if and only if it has the right lifting property with respect to the maps in I.

2. A map is a fibration if and only if it has the right lifting property with respect to the maps in J.

The sets I and J are called the *generating cofibrations* and *generating trivial cofibrations*, respectively.

In a cofibrantly generated model category, there is the following characterization of the fibrations, cofibrations, trivial cofibrations and trivial fibrations.

Proposition 1.2.5 ([Hir03, Prop. 11.2.1]). *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Then*

1. *the cofibrations of $\mathcal{M}$ are the retracts of relative I-cell complexes, which equals the class of I-cofibrations,*

2. *the class of trivial fibrations in $\mathcal{M}$ is the class of I-injectives,*

3. *the trivial cofibrations of $\mathcal{M}$ are the retracts of relative J-cell complexes, which equals the class of J-cofibrations,*

4. *the class of fibrations in $\mathcal{M}$ is the class of J-injectives.*

Example 1.2.6. Let R be a commutative ring. We wish to give the category $\mathbf{Ch}(R)$ of chain complexes of R-modules the structure of a cofibrantly generated model category.

Denote by S^n the chain complex with R in the n-th degree, and 0 in all other degrees, and let D^n be the chain complex with R in degrees n and $n-1$, and 0 otherwise, with the identity as the n-th differential. Let I be the set of inclusions $S^{n-1} \to D^n$ and J the set of inclusions $0 \to D^n$. We say that a map $f : X \to Y$ of chain complexes is a *quasi-isomorphism* if it induces an isomorphism on homology.

This setup makes $\mathbf{Ch}(R)$ into a cofibrantly generated model category, with I as the set of generating cofibrantions and J the set of generating trivial cofibrations. The proof given in [Hov01] relies on a more general recognition principle for cofibrant model categories.

By Prop. 1.2.5, the fibrations in $\mathbf{Ch}(R)$ are maps with the left lifting property with respect to J. Hence, a map $f : X \to Y$ of chain complexes is a fibration if for every n and map $R \to Y$, there is a lift in the diagram

$$\begin{tikzcd} & X_n \arrow[d, "f_n"] \\ R \arrow[r] \arrow[ur, dashed] & Y_n \end{tikzcd}$$

As a map $R \to Y_n$ is the same as a picking an element in Y_n, this means that f is a fibration if and only if it is surjective in each degree. Consequently, every object is fibrant.

Describing the cofibrant objects is somewhat more complicated, but it is a fact that every cofibrant objective is projective in each degree. As a consequence, if M is a R-module then taking the cofibrant approximation of the chain complex with M in degree zero and 0 in all other degrees gives a projective resolution of M by passing to homology.

Example 1.2.7. The model structure on **SSet** in Example 1.1.4 is cofibrantly generated. The set of generating cofibrations is the set

$$I = \{\delta\Delta^n \hookrightarrow \Delta^n : n \geq 0\}$$

of inclusions of faces into the standard n-simplexes, and the set of generating acyclic cofibrations is the set

$$J = \left\{\Lambda^k_n \hookrightarrow \Delta^n : n \geq 0\right\}$$

of inclusions of k-horns into the standard n-simplexes.

We will see in Sec. 1.2.5 that objects in $sPre(\mathcal{C})$ can be tensored with simplicial sets. Using this tensor product, the projective model structure on $sPre(\mathcal{C})$ is cofibrantly generated with generating cofibrations

$$I = \{\delta\Delta^n \otimes X \to \Delta^n \otimes X : n \geq 0, X \in \mathcal{C}\}$$

and generating trivial cofibrations

$$J = \left\{\Lambda^k_n \otimes X \to \Delta^n \otimes X : n \geq 0, X \in \mathcal{C}\right\}.$$

1.2.4 Compact objects and cellular model categories

The purpose of this section is to introduce cellular model categories. The definitions are cumbersome, but they are necessary in Hirschhorn's localization theory. The reader may safely skip this section on a first reading. The section also contains some conflicting terminology, in that the definition of compact objects used by Hirschhorn in order to define a cellular model category does not coincide with the definition of compact objects in triangulated categories. The definition of a compact object given below will not be used outside this section, and we hope this will not cause any confusion.

If $f : X \to Y$ is a relative I-cell complex, then a *presentation of* f is a λ-sequence

$$X = X_0 \to \cdots \to X_\beta \to \cdots \ (\beta < \lambda)$$

and a triple

$$\left\{(T^\beta, e^\beta, h^\beta)\right\}_{\beta<\lambda}$$

such that

1. the composition of the λ-sequence is isomorphic to f,
2. for every $\beta < \lambda$,
   - T^β is a set,
   - e^β is a function $T^\beta \to I$,
   - if $i \in T^\beta$ and $e^\beta(i)$ is the map $C_i \to D_i$ in I, then h^β is the disjoint union of maps $h_i^\beta : C_i \to X_\beta$ such that $X_{\beta+1}$ is the pushout of the diagram

$$\begin{tikzcd} \coprod_{T^\beta} C_i \arrow[r] \arrow[d, "\coprod h_i^\beta"'] & \coprod_{T^\beta} D_i \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

A *presented relative I-cell complex* is a relative cell complex and a particular presentation $\left\{(T^\beta, e^\beta, h^\beta)\right\}_{\beta<\lambda}$. Given such a presented relative I-cell complex f, then λ is said to be the *presentation ordinal of* f. The set $\coprod_{\beta<\lambda} T^\beta$ is the *set of cells of* f and the *size of* f is the cardinal of the set of cells of f. If $\beta < \lambda$, then X_β is the β-skeleton of f.

Example 1.2.8. Let $I = \left\{S^{n-1} \hookrightarrow D^n : n \geq 0\right\}$ be the set of inclusions of the n-spheres into the n-disk, with the convention that $S^{-1} = \emptyset$. Then a CW-complex X is an I-cell complex of finite size if X is finite (as a CW-complex) or ω if X is an infite CW-complex. Letting T^n be the set of n-cells for every $n \geq 0$, e^n be the maps $\alpha \mapsto (S^{n-1} \hookrightarrow D^n)$ and h_α^n be the attaching maps of the n-cells gives a presentation of X.

Remark. The above example is slightly misleading, because not every I-cell complex of topological spaces is a CW-complex. Recall that the for a CW-complex the image of the attaching map of a n-cell must be of cells in lower dimensions, which corresponds to the maps e^n above. This need not be true in general, however, and there is neither any reason that the size of an n-cell complex should be less than or equal to ω.

If $f : X \to Y$ is a presented relative I-cell complex with presentation $\left\{(T^\beta, e^\beta, h^\beta)_{\beta<\lambda}\right\}$, then a *subcomplex* of f is a presented relative I-cell complex $\tilde{f} : X \to \tilde{Y}$ with presentation $\left\{(\tilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta)_{\beta<\lambda}\right\}$ such that

1. for every $\beta < \lambda$, the set $\tilde{T}^\beta$ is a subset of T^β,

2. there is a map of λ-sequences such that for every ordinal β with $\beta + 1 < \lambda$, the diagram

$$\begin{tikzcd} \tilde{X}_\beta \arrow[r] \arrow[d] & \tilde{X}_{\beta+1} \arrow[d] \\ X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

   commutes, and for every $i \in \tilde{T}^\beta$, the map $h_i^\beta : C_i \to X_\beta$ factors as the composition of $\tilde{h}_i^\beta$ and the map $\tilde{X}^\beta \to X^\beta$.

If γ is a cardinal, then an object W is *γ-compact* if for every presented relative I-cell complex $f : X \to Y$, any map $W \to Y$ factors through a subcomplex of f of size at most γ. An object is *compact* if it is γ-compact for some ordinal γ.

In an arbitrary category, a map $A \to B$ is said to be an *effective monomorphism* if it is the equalizer of the two inclusion maps $B \rightrightarrows B \coprod_A B$.

Definition 1.2.9. A *cellular model category* is a cofibrantly generated model category $\mathcal{M}$ with generating cofibrations I and generating trivial cofibrations J such that

1. the domains and codomains of the elements of I are compact,

2. the domains of J are small relative to I,

3. the cofibrations are effective monomorphisms.

1.2.5 Simplicial model categories

Definition 1.2.10. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a *simplicial category* if there exists a functor $\underline{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{SSet}$ such that whenever X, Y are objects in $\mathcal{C}$, then

1. $\underline{\mathcal{C}}(X,Y)_0 = \mathcal{C}(X,Y)$

2. The functor $\underline{\mathcal{C}}(X,-) : \mathcal{C} \to \mathbf{SSet}$ has a left adjoint

$$X \otimes - : \mathbf{SSet} \to \mathcal{C}$$

   such that for two simplicial sets K and L, there is an isomorphism

$$X \otimes (K \times L) \cong (X \otimes K) \otimes L.$$

3. The functor $\underline{\mathcal{C}}(-,Y) : \mathcal{C}^{op} \to \mathbf{SSet}$ has a left adjoint functor $\mathbf{SSet} \to \mathcal{C}^{op}$.

Let $\mathcal{M}$ be a model category which is also a simplicial category, and let $i : A \to B$ be a cofibration and $p : X \to Y$ a fibration. Then $\mathcal{M}$ is a *simplicial model category* if the map

$$\underline{\mathcal{C}}(B,X) \xrightarrow{(i^*,p_*)} \underline{\mathcal{C}}(A,X) \times_{\underline{\mathcal{C}}(A,Y)} \underline{\mathcal{C}}(B,Y)$$

is a fibration of simplicial sets, which is trivial if either i or p is trivial.

The first example of a simplicial model category is the category of simplicial sets itself. Using the following lemma, which appears as Lemma II.2.4 in [GJ09], we get another example.

Lemma 1.2.11. *Let $\mathcal{C}$ be a category equipped with a functor $- \otimes - : \mathcal{C} \times \mathbf{SSet} \to \mathcal{C}$ such that the following conditions hold.*

1. *For fixed $K \in \mathbf{SSet}$, the functor $- \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\mathbf{hom}(K, -)$.*
2. *For fixed $A \in \mathcal{C}$, the functor $A \otimes - : \mathbf{SSet} \to \mathcal{C}$ commutes with arbitrary colimits, and $A \otimes * \cong A$.*
3. *There is an isomorphism $A \otimes (K \times L) \cong (A \otimes K) \otimes L$ for $A \in \mathcal{C}$ and $K, L \in \mathbf{SSet}$.*

Then $\mathcal{C}$ is a simplicial category with $\underline{\mathcal{C}}(A, B)$ defined by

$$\underline{\mathcal{C}}(A, B)_n = \mathcal{C}(A \otimes \Delta^n, B)$$

Example 1.2.12. The model structures in Example 1.1.6 for simplicial presheaves on a category $\mathcal{C}$ can be given the structure of a simplicial model category.

Define a tensor product by letting $X \otimes K$ be the presheaf given by $U \mapsto X(U) \times K$. The right adjoint $\mathbf{hom}(K, -)$ of $-\otimes K$ is defined by letting $\mathbf{hom}(K, X)$ be the presheaf $U \mapsto \underline{\mathbf{SSet}}(K, X(U))$. Hence, by Lemma 1.2.11, $sPre(\mathcal{C})$ is a simplicial category.

By a similar argument, the smash product makes the category $sPre(\mathcal{C})_\bullet$ of pointed simplicial presheaves has a tensor action and coaction from $\mathbf{SSet}_\bullet$ induced by the smash product.

1.2.6 The flasque model structure on $sPre(\mathcal{C})$

Let $\mathcal{C}$ be a category. We have already seen that $sPre(\mathcal{C})$, the category of simplicial presheaves on $\mathcal{C}$, forms a simplicial model category. In fact, we have seen two model structures on $sPre(\mathcal{C})$.

There is a third model structure which is Quillen equivalent to the projective and injective model structures. The model structure is cofibrantly generated, and it has some technical properties that will be convenient for our purposes.

Note that if C is an object of $\mathcal{C}$, then the Yoneda embedding $h^C : \mathcal{C}^{op} \to \mathbf{Set}$ given by $C \mapsto \mathrm{Hom}(-, C)$ extends to a simplicial presheaf of dimension 0. Any simplicial set K gives rise to a constant simplicial presheaf.

Definition 1.2.13. Let X be an object of $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to X\}$ be a finite collection of monomorphisms. Define the *union* $\cup\mathcal{U}$ of $\mathcal{U}$, to be the coequalizer to the diagram

$$\coprod_{i,j} U_i \times U_j \rightrightarrows \coprod_i U_i$$

in $sPre(\mathcal{C})$. There is an induced map $\cup\mathcal{U} \to X$. Any such map is called *acceptable*.

Definition 1.2.14. Let $f : X \to Y$ be a map of simplicial presheaves on $\mathcal{C}$ and $g : K \to L$ a map of simplicial sets. Define the pushout product $f \Box g$ to be the induced map

$$X \otimes L \cup_{X \otimes K} Y \otimes K \to Y \otimes L$$

in the pushout diagram.

$$\begin{tikzcd}
X \otimes K \arrow[r, "f \otimes \mathrm{id}_K"] \arrow[d, "\mathrm{id}_X \otimes g"] & Y \otimes K \arrow[d, "\mathrm{id}_Y \otimes g"] & \\
X \otimes L \arrow[r, "f \otimes \mathrm{id}_L"] & X \otimes L \cup_{X \otimes K} Y \otimes K \arrow[dr, "f \Box g"] & \\
& & Y \otimes L
\end{tikzcd}$$

Definition 1.2.15. Let

a) I_{fl} be the set of pushout products $f \Box i$ where f an acceptable map, and $i : \partial\Delta^n \to \Delta^n$ is a generating cofibration of simplicial sets,

b) J_{fl} be the set of pushout products $f \Box j$ where f an acceptable map and $j : \Lambda^n_k \to \Delta^n$ is a generating acyclic cofibration of simplicial sets.

Definition 1.2.16. Let $\mathcal{C}$ be a category.

a) A map f in $sPre(\mathcal{C})$ is a flasque fibration if it is that are J_{fl}-injective, i.e. have the right lifting property with respect to all maps in J_{fl}.

b) A map f in $sPre(\mathcal{C})$ is a flasque cofibration if it is that are J_{fl}-projective, i.e. have the left lifting property with respect to all maps in J_{fl}.

Theorem 1.2.17 ([Isa05, Thm. 3.7])**.** *Let $\mathcal{C}$ be a category.*

a) The objectwise weak equivalences, flasque cofibrations and flasque fibrations form a proper, cellular model structure. The maps in I_{fl} and J_{fl} are generating cofibrations and acyclic cofibrations for the model structure.

b) The identity functor is a left Quillen equivalence from the injective model structure to the flasque model structure and from the flasque model structure to the projective model structure.

c) If $\mathcal{C}$ contains finite products, then the flasque model structure is simplicial.

1.3 Bousfield localization

Let $\mathcal{M}$ be a simplicial model category, and $\mathcal{C}$ a class of maps in $\mathcal{M}$. In analogy with localization of a ring in a multiplicatively closed subset, we wish to be able to localize the model structure of $\mathcal{M}$ in $\mathcal{C}$, such that the maps in $\mathcal{C}$ becomes weak equivalences. An object X in $\mathcal{M}$ is *$\mathcal{C}$-local* if it is fibrant and for every map $f : A \to B$ the induced map of simplicial mapping complexes, $f^* : \underline{\mathcal{M}}(B, X) \to \underline{\mathcal{M}}(A, X)$, is a weak equivalence (of simplicial sets). A map $f : X \to Y$ in $\mathcal{M}$ is a *$\mathcal{C}$-local equivalence* if for every $\mathcal{C}$-local object W the induced map of simplicial mapping complexes $f^* : \underline{\mathcal{M}}(B, X) \to \underline{\mathcal{M}}(A, X)$ is a weak equivalence.

Remark. In particular, the definition of $\mathcal{C}$-local objects implies that every map in $\mathcal{C}$ is a $\mathcal{C}$-local equivalence. Furthermore, in a simplicial model category, a weak equivalence $f : A \to B$ induces a weak equivalence $f^* : \underline{\mathcal{M}}(B, W) \to \underline{\mathcal{M}}(A, W)$ for every fibrant object W, so the weak equivalences of $\mathcal{M}$ are $\mathcal{C}$-local equivalences for any class $\mathcal{C}$.

Definition 1.3.1. Let $\mathcal{M}$ be a simplicial model category and $\mathcal{C}$ a class of maps in $\mathcal{M}$. Suppose there exists a model structure $L_{\mathcal{C}}\mathcal{M}$ such that

1. the class of weak equivalences of $L_{\mathcal{C}}\mathcal{M}$ are the $\mathcal{C}$-local equivalences of $\mathcal{M}$,
2. the class of cofibrations of $L_{\mathcal{C}}\mathcal{M}$ are the cofibrations of $\mathcal{M}$,
3. the class of fibrations of $L_{\mathcal{C}}\mathcal{M}$ are the the maps that have the right lifting property with respect to maps that are both cofibrations and $\mathcal{C}$-local equivalences.

Then $L_{\mathcal{C}}\mathcal{M}$ is the *left Bousfield localization* of $\mathcal{M}$ with respect to $\mathcal{C}$.

As an immediate consequence, note that if $\mathcal{M}$ is cofibrantly generated with generating cofibrations I, then I is a set of generating cofibrations for $L_{\mathcal{C}}\mathcal{M}$, if it exists. Furthermore, the fibrant objects of $L_{\mathcal{C}}\mathcal{M}$ are precisely the $\mathcal{C}$-local objects.

Whenever the Bousfield localization exists, it has the following properties.

Proposition 1.3.2. *1. If $L_{\mathcal{C}}\mathcal{M}$ exists, and $F : \mathcal{M} \to \mathcal{N}$ is a Quillen functor which takes every cofibrant approximation of a map in $\mathcal{C}$ to a weak equivalence in $\mathcal{N}$, then F is a Quillen functor considered as a functor $L_{\mathcal{C}}\mathcal{M}$.*

2. If $F : \mathcal{M} \to \mathcal{N}$ is a Quillen equivalence, then it induces a Quillen equivalence as a functor $F : L_{\mathcal{C}}\mathcal{M} \to L_{\mathbb{L}F\mathcal{C}}\mathcal{N}$.

If $\mathcal{M}$ is an arbitrary simplicial model category and $\mathcal{C}$ is an arbitrary class of maps in $\mathcal{M}$, there is no guarantee that the left Bousfield localization with respect to $\mathcal{C}$ exists. However, the following theorem, which is the main theorem in [Hir03], implies that it will exist in the cases we will consider.

Theorem 1.3.3 ([Hir03, Theorem 4.1.1])**.** *Let $\mathcal{M}$ be a simplicial, left proper cellular model category and S a set of maps in $\mathcal{M}$. Then*

1. *the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ with respect to S exists,*
2. *$L_S\mathcal{M}$ is left proper and cellular,*
3. *the simplicial structure on $\mathcal{M}$ gives $L_S\mathcal{M}$ the structure of a simplicial model category.*

1.4 Spectra

In algebraic topology important invariants such as (reduced) ordinary (co)homology, topological K-theory and stable homotopy groups share a common feature, they are in some sense stable under application of the suspension functor. An example of this is Freudenthal's suspension theorem [May99], which states that if X is an $(n-1)$-connected cofibrant based topological space, then the suspension functor induces an isomorphism $\pi_n(X) \cong \pi_{n+1}(\Sigma X)$ of homotopy groups.

In the following, let $\mathcal{M}$ be a cofibrantly generated model category, and $G : \mathcal{M} \to \mathcal{M}$ a Quillen functor. A *G-spectrum* is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of objects in $\mathcal{M}$ together with *structure maps* $\sigma : GX_n \to X_{n+1}$. A map of G-spectra $f : X \to Y$ is a sequence of maps $f_n : X_n \to Y_n$ compatible with the structure maps, so that for every n, the diagram commutes.

$$\begin{tikzcd} GX_n \arrow[r, "\sigma"] \arrow[d, "Gf_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ GY_n \arrow[r, "\sigma"] & X_{n+1} \end{tikzcd}$$

It follows that there is a category of G-spectra, which we denote $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$. For any functor $T : \mathcal{M} \to \mathcal{M}$ and natural transformation $\tau : GH \to HG$ there is an extension of H to $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ by letting $H(X)_n = H(X_n)$ with structure maps $GH(X_n) \xrightarrow{\tau} H(GX_n) \xrightarrow{T\sigma} H(X_{n+1})$. We call this extension the *prolongation* of H. In particular, the identity transformation gives prolongation of G to $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$. We do not make any notational distinction bewteen G and its prolongation.

Example 1.4.1 (Cf. [BF78])**.** Let $\mathcal{M}$ be the category of pointed simplicial sets and let $G = \Sigma$ be given by smashing with S^1. There is a functor $\Sigma^\infty : \mathbf{SSet}_\bullet \to \mathrm{Sp}^{\mathbb{N}}(\mathbf{SSet}_\bullet, \Sigma)$ given by $(\Sigma^\infty X)_n = \Sigma^n X$ and letting all structure maps be given by the twist map. We can define homotopy groups for spectra by letting $\pi_q^s(X) = \mathrm{colim}_n\, \pi_{n+q}(|X_n|)$, where the latter are the ordinary homotopy groups. If X is a simplicial sets it follows that $\pi_q^s(\Sigma^\infty X_+)$ is the q-th stable homotopy group of X.

Remark. There is a subtle difference between our prolongation of Σ and Bousfield and Friedlander's classical construction on **SSet** with the suspension functor [BF78] in that they use the twist map on $S^1 \wedge S^1$ to prolongate Σ. In our more general case, this is not necessarily possible.

We wish to construct a model structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ such that G becomes a Quillen equivalence. This can be done by going through an intermediate model structure, which we will call the *strict model structure*. A map $f : X \to Y$ of spectra will be said to be a *strict weak equivalence* if every $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{M}$, and we define *strict fibrations* in the same manner. In light of Proposition 1.1.2, the *strict cofibrations* are defined to be the maps that have the left lifting property with respect to strict trivial fibrations.

Theorem 1.4.2 ([Hov01, Thm. 1.14])**.** *The strict fibrations, cofibrations and weak equivalences define a cofibrantly generated model structure on $Sp^{\mathbb{N}}(\mathcal{M}, G)$. The model structure is left proper if the model structure on $\mathcal{M}$ is left proper, and it is cellular if the model structure on $\mathcal{M}$ is. We call this model structure the* strict model structure*.*

Assuming $\mathcal{M}$ is left proper and cellular, we can therefore apply the technique of Bousfield localization.

In order to make G into a Quillen equivalence, we define a functor $F_n : \mathcal{M} \to \mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ by

$$(F_n(X))_m = \begin{cases} G^{n-m}X & \text{if } n \geq m, \\ 0 & \text{if } n < m. \end{cases}$$

Remark. We will use the notation F_n^G if we wish to emphasize the functor G.

If Q is a cofibrant replacement functor for $\mathcal{M}$, define the set

$$S = \left\{ F_{n+1}GQC \xrightarrow{s_n^{QC}} F_nQC \right\},$$

where C runs over the domains and codomains of the generating cofibrations, and s_n^{QC} is the adjoint of the identity on GQC. Define the *stable model structure* on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ to be the Bousfield localization of the strict model structure with respect to S.

Theorem 1.4.3 ([Hov01, Thm. 3.8])**.** *If $\mathcal{M}$ is left proper and cellular, then $G : Sp^{\mathbb{N}}(\mathcal{M}, G) \to Sp^{\mathbb{N}}(\mathcal{M}, G)$ and the shift functor $t : Sp^{\mathbb{N}}(\mathcal{M}, G) \to Sp^{\mathbb{N}}(\mathcal{M}, G)$ are Quillen equivalences with respect to the stable model structure.*

As we have seen in Ex. 1.1.15 and Sec. 1.2.6, there are several Quillen equivalent model structures on $sPre(\mathcal{C})$. If $G : sPre(\mathcal{C}) \to sPre(\mathcal{C})$ is a Quillen endofunctor in all these model structures, we would like there to be a Quillen equivalence between the induced stable model categories of G-spectra. This is achieved in the following results.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are left proper cellular model categories, G is a left Quillen endofunctor of $\mathcal{C}$ and H is a left Quillen endofunctor of $\mathcal{D}$. A *map of pairs* $(\phi, \tau) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ is a left Quillen functor $\phi : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\tau : \phi G \to H\phi$ such that τ_A is a weak equivalence for all cofibrant A in $\mathcal{C}$.

Proposition 1.4.4 ([Hov01, Prop 5.3]). *Suppose $(\phi, \tau) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ is a map of pairs. Then there is an induced map of pairs $(Sp^{\phi}, Sp^{\tau}) : Sp(\mathcal{C}, G) \to Sp(\mathcal{D}, H)$ such that $Sp^{\phi} \circ F_n^G = F_n^H \circ \phi$. This induced map is compatible with composition and identities.*

Theorem 1.4.5 ([Hov01, Thm. 5.5]). *Suppose $(\phi, \tau) : (\mathcal{C}, G) \to (\mathcal{D}, H)$ is a map of pairs such that Φ is a Quillen equivalence and τ_X is a weak equivalence for all $X \in \mathcal{M}$. Then the induced Quillen functor Sp^{Φ} is a Quillen equivalence.*

Finally, we will need some results in the case where the model category $\mathcal{M}$ also has a symmetric monoidal structure. We would like this to extend to $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$, but unfortunately, this is not possible in general.

Theorem 1.4.6 ([Hov01, Thm. 5.7]). *Assume $\mathcal{M}$ has a symmetrical monoidal structure and there is a coherent isomorphism $G(X \otimes K) \cong GX \otimes K$ for $X, Y \in \mathcal{M}$. Then $Sp^{\mathbb{N}}(\mathcal{M}, G)$ is tensored, cotensored and enriched over $\mathcal{M}$, compatibly with the model structure.*

The action of $\mathcal{M}$ on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ is given levelwise by defining, for $X \in \mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ and $K \in \mathcal{M}$, by

$$G(X \otimes K)_n \cong GX_n \otimes K \xrightarrow{\sigma \otimes \mathrm{id}} X_{n+1}.$$

The construction of the cotensor is constructed similarily.

Suppose that $\mathcal{M}$ has a symmetrical monoidal structure, and $G = - \otimes K$ for some object K in $\mathcal{M}$. As the isomorphism $G(X \otimes K) \cong GX \otimes X$ is given by the twist map, it follows that the tensor product is different from the prolongation of G, where the structure maps do not involve a twist. Hence, we have to ways of tensoring with K, and we denote the tensor given from the prolongation by $X \mapsto X \overline{\otimes} K$. Although we already know that this latter map is a Quillen equivalence, it does not follow that $X \mapsto X \otimes K$ is a Quillen equivalence.

Remark. In the case that $\mathcal{M}$ is symmetrical monoidal and G is given by tensoring by an object K, we denote the category of G-spectra by $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$.

Moreover, the action of $\mathcal{M}$ on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$ does not necessarily give $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$ the structure of a monoidal category, as can be seen by the following example.

Example 1.4.7 (Cf. [Hov01, Lemma 5.10]). If $\mathcal{M}$ is a symmetric monoidal category with unit S, the tensor product gives the category $\mathcal{M}^{\mathbb{N}}$ of sequences in $\mathcal{M}$ a symmetrical monoidal structure by

$$(X \otimes Y)_n = \coprod_{p+q=n} X_p \otimes Y_q.$$

The object $T = (K^{\otimes n})_{n \geq 0}$ is a monoid object in $\mathcal{M}^{\mathbb{N}}$, meaning that for any $X \in \mathcal{M}^{\mathbb{N}}$, the set $\mathcal{M}T, X$. Then the category $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$ is the subcategory of left T-modules in $\mathcal{M}^{\mathbb{N}}$. However, the monoid T is not commutative unless the commutativity isomorphism on $K \otimes K$ in $\mathcal{M}$ is the identity. If this is not the case, then $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$ is not symmetric.

To remedy this, Hovey [Hov01] shows how to construct a model category of symmetric spectra, $\mathrm{Sp}^{\Sigma}(\mathcal{M}, K)$. This construction is slightly more complicated than the one of $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, K)$, but it is preferable in some applications because of its symmetric monoidal structure. However, Hovey also shows that if the object K that defines the suspension functor is *symmetric*, meaning that the cyclic permuation of $K \otimes K \otimes K$ is the identity, there is a model category $\mathcal{E}$ and a zig-zag of Quillen equivalences $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G) \to \mathcal{E} \leftarrow \mathrm{Sp}^{\Sigma}(\mathcal{M}, K)$. Hence, there is an equivalence of the homotopy categories of $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}, G)$ and $\mathrm{Sp}^{\Sigma}(\mathcal{M}, K)$. This result is a corollary of the following theorem.

Theorem 1.4.8 ([Hov01, Thm. 9.3]). *Suppose the cofibrantly generated model category $\mathcal{M}$ is left proper, cellular and symmetric monoidal with cofibrant unit. Assume either K is symmetric and cofibrant, or that the domains of the generating cofibrations of M is cofibrant and K is weakly equivalent to a symmetric object of $\mathcal{M}$. Then the functor $X \mapsto X \otimes K$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathcal{M}, K)$.*

Remark. As we know from Thm. 1.4.6 that the tensor product is compatible with the model structure, the left derived functor of the tensor product gives a tensor product on $\operatorname{Ho}\operatorname{Sp}^{\mathbb{N}}(\mathcal{M}, K)$. Thm. 1.4.8 is therefore equivalent to saying that the functor $X \mapsto X \otimes K$ gives a self-equivalence on $\operatorname{Ho}\operatorname{Sp}^{\mathbb{N}}(\mathcal{M}, K)$.

Chapter 2

Equivariant motivic spaces

Having discussed model categories, we know turn our attention to equivariant motivic homotopy theory. As categories of schemes generally do not have the nice properties we need to have a model category, we will use the simplicial presheaves to embed our category into a model category. First, however, we will need discuss Grothendieck topologies in order to take account of some of the "local" data in algebraic topology. A reference for Grothendieck can be found in [Fan+05]. In our discussion of equivariant motivic homotopy theory, we will mostly follow [HKØ14].

2.1 Grothendieck topologies

A sheaf on a topological space X is a contravariant functor from the category of open subsets of X to **Set** (or some other category), satisfying some "local" compatibility conditions. These conditions can be specified by describing how sheaves should behave on coverings. Explicitly, if $\mathcal{F}$ is a sheaf on X and $\{U_\alpha\}$ is an open cover of the open set U, then the sequence

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_\alpha \times_X U_\beta)$$

is an equalizer. As a consequence, in order to check if a presheaf is a sheaf it suffices to know how the presheaf behaves on coverings. Grothendieck topologies abstracts the notion of a covering from topological spaces to more general categories.

Let $\mathcal{C}$ be a small category. A *Grothendieck pre-topology on* $\mathcal{C}$ consists of set $\mathrm{Cov}(X)$ of *covering families* for each object X in $\mathcal{C}$, that is, sets $\{f_\alpha : U_\alpha \to X : \alpha \in A\}$ of maps for every object X in $\mathcal{C}$ such that:

GT1 For every object X in $\mathcal{C}$, the set $\{\mathrm{id}_X\}$ is an element of $\mathrm{Cov}(X)$.

GT2 If $\{f_\alpha : U_\alpha \to X : \alpha \in A\}$ is a covering family in $\mathrm{Cov}(X)$ and $f : Y \to X$ is a map such that the fiber products $Y \times_X U_\alpha$ exists for each $\alpha \in A$, then the family

$$\{Y \times_X U_\alpha \to Y : \alpha \in A\}$$

is a covering family in $\mathrm{Cov}(Y)$.

GT3 If $\{f_\alpha : U_\alpha \to X : \alpha \in A\}$ is a covering family of X, and for each $\alpha \in A$ there is a covering family $\{g_\beta : V_{\alpha\beta} \to U_\alpha : \beta \in B\}$ of U_α, then $\{g_{\alpha\beta} \circ f_\alpha : V_{\alpha\beta} \to X\}$ is an element of $\mathrm{Cov}(X)$.

A category equipped with a Grothendieck pre-topology is called a *site*.

Remark. As the name pre-topology suggests, there is also a notion of a Grothendieck topology for a small category. However, we will not need it in the following.

Example 2.1.1. The most basic example of a Grothendieck pre-topology is the category $\mathrm{Op}(X)$ of open subsets of a topological space X, where the maps are inclusions. The covering families are simply open covers. In this category, fibered products are just intersections, so (GT2) amounts to saying that if $\{U_\alpha : \alpha \in A\}$ is an open cover of U and V is an open subset of U, then $\{U_\alpha \cap V : \alpha \in A\}$ is an open cover of V.

In the case where $X \in \mathbf{Sch/S}$ is a scheme of finite type over a Noetherian base scheme S, the category $\mathrm{Op}(|X|)$ on its underlying topological space with this Grothendieck pre-topology is called the *(small) Zariski site*, denoted X_{Zar}.

Example 2.1.2. A scheme X over a base scheme S gives rise to other Grothendieck pre-topologies. The *étale site*, $X_{ét}$, is defined by letting the covering families consist of étale coverings of schemes over X, meaning that a covering family is a set of maps $\{f_\alpha : U_\alpha \to U : \alpha \in A\}$, where each f_α is an étale map, and U is a scheme over X.

2.1.1 Grothendieck topologies generated by cd-structures

A way to generate Grothendieck topologies on a category is the machinery of cd-structures [Voe10]. Suppose $\mathcal{D}$ is a collection of commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ B \arrow[r, "i"] & X \end{tikzcd} \tag{2.1}$$

in some category C with an initial object. We say that $\mathcal{D}$ is a *cd-structure* if it is closed under isomorphisms of squares. If $\mathcal{D}$ is a cd-structure, then the squares (2.1) are called *distinguished squares.* The Grothendieck topology on C associated to a cd-structure $\mathcal{D}$ is the smallest Grothendieck topology on $\mathcal{C}$ such that the for every square (2.1), the maps p and i are covering.

If $\mathcal{D}$ is a cd-structure on a category $\mathcal{C}$, then the Grothendieck topology generated by $\mathcal{D}$ is the smallest Grothendieck topology on $\mathcal{C}$ such that for every distinguished square 2.1 the set $\{p : Y \to X, i : B \to X\}$ is a covering familiy.

2.2 The Nisnevich Topology

2.2.1 Group schemes

Let S be a fixed separated Noetherian scheme of finite Krull dimension. Recall that a *group scheme over* S is a scheme G together with a section $\varepsilon : S \to G$ and maps $\mu : G \times G \to G$ and $\iota : G \to G$, called the *unit*, *multiplication* and *inverse* such that the following diagrams commute.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id},\ \iota)"] \arrow[d] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(\iota,\ \mathrm{id})"'] \arrow[d] \\ S \arrow[r, "\varepsilon"] & G & S \arrow[l, "\varepsilon"] \end{tikzcd}$$

$$\begin{tikzcd} G \times S \arrow[r, "\mathrm{id} \times \varepsilon"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & G \times S \arrow[l, "\varepsilon \times \mathrm{id}"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

We will also impose the condition that G is separated and of finite type over S.

Remark. It is not necessarily true that the set of points of a group scheme G form a group. However, a group scheme is a group object in the category of schemes. It follows that for any group scheme G and scheme X, the set $\mathrm{Hom}_S(X, G)$, i.e. the X-valued points of G, carries the structure of a group. This suggests an alternative definition: a group scheme over S is a representabe functor $\mathbf{Sch/S} \to \mathbf{Set}$ which factors through $\mathbf{Grp}$.

Example 2.2.1. Suppose $S = \mathrm{Spec}(R)$ for some ring R and let $A = R[x, x^{-1}]$. Applying Spec to the R-algebra maps $m : A \to A \otimes A$ defined by $x \mapsto x \otimes x$, $i : A \to A$ defined by $x \mapsto x^{-1}$ and $u : A \to R$ defined by $x \mapsto 1$ gives $\mathrm{Spec}(A)$ the structure of a group scheme, the multiplicative group over R, which we denote $\mathbb{G}_m$.

Example 2.2.2. Let G be a finite group. Let $\Gamma = \coprod_{g \in G} S_g$, that is, the disjoint union of copies of S labeled by the elements of G. A map $\Gamma \times_S \Gamma \to \Gamma$ is the same thing as one map $S \to \Gamma$ for each pair $(g_1, g_2) \in G \times G$. Furthermore, each such map must factor as the identity and the inclusion of one of the copies of S into Γ. Define the multiplication map $\mu : \Gamma \times_S \Gamma \to \Gamma$ as the map which sends the copy of S labeled (g_1, g_2) to the one labeled $g_1 g_2$, and the inverse map as the map sending the copy labeled g_1 to the one labeled g_1^{-1}. The unit map $S \to \Gamma$ is the inclusion of the copy labeled by the identity element of G. This turns Γ into a group scheme with $|\Gamma| \cong G$. We say that Γ is the *finite, constant group scheme* G.

If G is a group scheme and X a scheme over S, a *group action of* G is a map $a : G \times X \to X$ which is compatible with the group structure on G, meaning that the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id} \times a"] \arrow[d, "\mu \times \mathrm{id}"] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"] & X \end{tikzcd}$$

commutes. A scheme equipped with a group action from a group scheme G is called a *G-scheme.* If X and Y are G-schemes, then an equivariant map $f : X \to Y$ is a map of schemes compatible with the G-scheme structures on X and Y:

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id} \times f"] \arrow[d, "a"] & G \times Y \arrow[d, "a"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If G is a fixed group scheme over S, then the class of G-schemes and equivariant maps form a category, which we will denote $G\mathbf{Sch/S}$. By requiring both the group actions and equivariant maps to be smooth, we also obtain a category of smooth G-schemes, $G\mathbf{Sm/S}$.

2.2.2 The Nisnevich topology for GSm/S

A *distinguished Nisnevich square* in $G\mathbf{Sm/S}$ is a commutative square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where p is an étale map, i an open embedding, and the induced map

$$p^{-1}(X \setminus U)_{red} \to (X \setminus U)_{red}$$

is an isomorphism. The collection of distinguished squares form a cd-structure on $G\mathbf{Sm/S}$, and the associated Grothendieck topology is called the *equivariant Nisnevich topology.*

There is another characterization of the covers in the equivariant Nisnevich topology, which was the definition originally proposed by Voevodsky [Del09, Sec. 3.1]. An equivariant map $f : X \to Y$ is said to have an *equivariant splitting sequence of length* n if there exists a filtration of invariant closed subschemes

$$\emptyset = Y_{n+1} \subseteq Y_n \subseteq \cdots \subseteq Y_0 = Y$$

such that, for every j, the induced map

$$(Y_j \setminus Y_{j+1}) \times_Y X \to (Y_j \setminus Y_{j+1})$$

has an equivariant section.

Proposition 2.2.3 ([HKØ14, Prop. 2.13])**.** *An equivariant étale map $X \xrightarrow{f} Y$ is an equivariant Nisnevich cover if and only if it has an equivariant splitting sequence.*

If G is a finite group, then there is a third description of the equivariant covers. For a G-scheme X and an element $x \in X$, the *set-theoretic* stabilizer S_x of x is the subgroup of G defined by

$$S_x = \{g \in G : gx = x\}.$$

Proposition 2.2.4 ([HKØ14, Prop. 2.17])**.** *If G is a finite group and $S = \mathrm{Spec}(k)$, where k is a field, then an equivariant étale map $f : X \to Y$ is an equivariant Nisnevich cover if and only if for every $y \in Y$ there is a $x \in X$ such that $f(x) = y$ and f induces isomorphisms $k(y) \cong k(x)$ and $S_x \cong S_y$.*

Remark. If G is trivial, we get the ordinary Nisnevich topology on $\mathbf{Sm/S}$, the category of smooth schemes over S.

Remark. Herrmann [Her13] discusses a variation of the Nisnevich topology. Assume G is a finite, constant group scheme and $S = \mathrm{Spec}(k)$, where k is a field. Define an equivariant étale map $f : X \to Y$ to be a *fixed point Nisnevich cover* if, for every subgroup $H \subseteq G$, the induced map on fixed points $f^H : X^H \to Y^H$ is a non-equivariant Nisnevich cover.

If X is a G-scheme and $x \in X$, define the *scheme-theoretic stabilizer of* x by the pullback diagram

$$\begin{array}{ccc} G_x & \longrightarrow & G \times X \\ \downarrow & & \downarrow (a, \mathrm{id}) \\ \mathrm{Spec}(k(x)) & \xrightarrow{\Delta \circ x} & X \times X \end{array}$$

An étale map $f : X \to Y$ is a fixed point Nisnevich cover if and only if for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$, and f induces isomorphisms on residue fields and scheme-theoretic stabilizers [Her]. Note that S_x is a pullback in the diagram

$$\begin{tikzcd} S_x \arrow[r] \arrow[d] & {|G| \times |X|} \arrow[d, "{(a, \mathrm{id})}"] \\ * \arrow[r, "\Delta \circ x"] & {|X| \times |X|} \end{tikzcd}$$

By the universal property of S_x, there is an induced map $|G_x| \to S_x$, and by inspection, this is an inclusion of subgroups. Hence, by Prop. 2.2.4, any equivariant Nisnevich cover is also a fixed point Nisnevich cover.

2.3 Local model structures for presheaves on a site

Let $\mathcal{C}$ be a fixed site. So far, we have seen how to construct model structures on $sPre(\mathcal{C})$, essentially exploiting the fact that simplicial preseheaves take values in **SSet**, which is a model category. If, however, $\mathcal{C}$ is a site, then the data of covering families in $\mathcal{C}$ provides "local" structure on $\mathcal{C}$, and there is no reason for the model structures we have seen so far to reflect any of this structure. A local model structure for $sPre(\mathcal{C})$ was first introduced by [Jar87]. As alluded to above, given a Grothendieck topology on $\mathcal{C}$, it is possible to define what a sheaf on $\mathcal{C}$ should be. In particular, it is possible to define sheaves of homotopy groups for ("locally" fibrant) simplicial presheaves. Jardine's model structure is constructed so that the weak equivalences are induced by isomorphisms of homotopy sheaves.

There are other ways of obtaining model structures on a site that are Quillen equivalent to Jardine's local model structures. One approach is introduced in [DHI04] using the machinery of hypercovers. In the case of the Nisnevich topology on $G\mathbf{Sm}/\mathbf{S}$, however, there is a way of constructing the local model stucture which will be more suitable for our purposes. It is shown in [Bla01] that this construction is Quillen equivalent to Jardine's.

For ease of notation, we will denote the category $sPre(G\mathbf{Sm}/\mathbf{S})$ of simplicial presheaves on the category of G-schemes over S by $\mathcal{M}^G(S)$. An object of $\mathcal{M}^G(S)$ will be called a *motivic G-space.* If Q is a distinguished equivariant Nisnevich square, let Q^{hp} denote its homotopy pushout in the global projective model structure. There is a natural map $Q^{hp} \to X$. Furthermore, let $\emptyset$ be the initial object in $sPre(G\mathbf{Sm}/\mathbf{S})$ and $h_\emptyset$ the Yoneda embedding of the empty G-scheme. Define

$$\Sigma^{hp} = \left\{Q^{hp} \to X\right\} \cup \{\emptyset \to h_\emptyset\}.$$

We define *the local projective (resp. flasque, resp. injective) model structure* on $sPre(G\mathbf{Sm}/\mathbf{S})$ to be the (left) Bousfield localization of the global projective (resp. flasque, resp. injective) model structure.

Theorem 2.3.1. *The local projective (resp. flasque, resp. injective) model structure on $\mathcal{M}^G(S)$ is cellular, proper, combinatorial and simplicial. The identity functor induces a left Quillen equivalence from the local projective (resp. local flasque) to the local flasque (resp. local injective) model structure.*

In the case of the local flasque and local injective model structures, the following proposition gives us a way of dispensing of the homotopy pushouts in the definition above.

Proposition 2.3.2 ([Isa05, Thm. 4.9])**.** *The local flasque (resp. local injective) model structures on $\mathcal{M}^G(S)$ is the Bousfield localization of the global flasque (resp. global injective) model structures in the set of maps*

$$U \coprod_{U \times_X V} V \to X$$

for distinguished equivariant Nisnevich squares.

2.4 The equivariant motivic model structures

We are know in a position to define the unstable and stable equivariant motivic model structures. For the remainder of the thesis, we assume that G is a finite constant group scheme.

2.4.1 The unstable model structure

The guiding intuition behind motivic homotopy theory is that the affine line should be contractible. We obtain this by defining the *equivariant motivic projective (resp. flasque, resp. injective) model structure* on $\mathcal{M}^G(S)$ to be the Bousfield localization of the local projective (resp. flasque, resp. injective) model structure in the set of maps

$$\left\{X \times_S \mathbb{A}^1_S \xrightarrow{p_X} X : X \in G\mathbf{Sm/S}\right\},$$

where $\mathbb{A}^1_S$ is given the trivial G-action. Since Bousfield localizations of Quillen equivalent model strucures in the same set of maps are Quillen equivalent, it follows that the weak equivalences in the equivariant motivic projective, flasque and injective model structures are the same. Hence we are justified in refering to them as *equivariant motivic weak equivalence.*

Theorem 2.4.1 ([HKØ14, Thm. 4.3])**.** *The equivariant motivic projective (resp. flasque, resp. injective) model structure on $\mathcal{M}^G(S)$ is cellular, proper and simplicial. The identity functor induces a left Quillen equivalence from the local projective (resp. local flasque) to the local flasque (resp. local injective) model structure.*

The pointed case is similar. We have a disjoint base point functor $(-)_+ : \mathcal{M}^G(S) \to \mathcal{M}^G_\bullet(S)$ which sends an unpointed presheaf $\mathcal{X}$ to $\mathcal{X} \coprod *$ pointed at the disjoint base point. The disjoint base point functor is left adjoint to the forgetful functor $\mathcal{M}^G_\bullet(S) \to \mathcal{M}^G(S)$. As in Example 1.1.7, a map of pointed presheaves $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ is a pointed equivariant motivic weak equivalence if and only if the map $f : \mathcal{X} \to \mathcal{Y}$ is an equivariant motivic weak equivalence. Thus, we can view the pointed equivariant motivic flasque model structure as the Bousfield localization of the pointed global flasque model structure in the set of maps

$$\mathcal{S} = \left\{(X \times_S \mathbb{A}^1_S \xrightarrow{p_X} X)_+ : X \in G\mathbf{Sm/S}\right\} \cup \left\{(U \coprod_{U \times_X V} V \to X)_+\right\},$$

where

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{\quad i \quad} & X \end{array}$$

is a distinguished Nisnevich square. To summarize, we have the following theorem.

Theorem 2.4.2 ([HKØ14, Thm. 4.6])**.** *The pointed equivariant motivic projective (resp. flasque, resp. injective) model structure is proper, cellular and simplicial. The identity induces Quillen equivalences between the projective, flasque and injective model structures.*

We denote the homotopy category associated with the pointed equivariant motivic model structure by $\mathrm{H}^G_\bullet(S)$.

In the pointed case, we can define a smash product on $\mathcal{M}^G_\bullet(S)$ schemewise.

Theorem 2.4.3 ([HKØ14, Thm. 4.7]). *The smash product gives $\mathcal{M}^G_\bullet(S)$ the structure of a symmetric monoidal category. Furthermore, the smash product preserves weak equivalences and injective cofibrations, and induces a symmetric monoidal structure on $H^G_\bullet(S)$.*

A nice result in [HKØ14] makes equivariant vector bundles invertible. An *elementary* $\mathbb{A}^1$-*homotopy* between maps $f, g : \mathcal{X} \to \mathcal{Y}$ of motivic G-spaces is a map $H : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{Y}$ such that $H \circ i_0 = h$ and $H \circ i_1 = g$. By construction, elementary $\mathbb{A}^1$-homotopic maps become equal in $\mathrm{H}^G(S)$.

Proposition 2.4.4 ([HKØ14, Thm. 4.7]). *Let $p : \mathcal{V} \to \mathcal{X}$ be a G-equivariant vector bundle. Then p is an equivariant motivic weak equivalence.*

Proof. Let $i : \mathcal{X} \to \mathcal{V}$ be the zero-section. As $p \circ i = \mathrm{id}$, it suffices to find an elementary $\mathbb{A}^1$-homotopy between $i \circ p$ and id.

Let $\mathcal{E} \to M$ be a vector bundle over the scheme M. Write $H_{\mathcal{E}} : \mathcal{E} \times \mathbb{A}^1 \to \mathcal{E}$ for the standard fiberwise contraction. On open affine subsets $U = \mathrm{Spec}(R) \subseteq M$ on which $\mathcal{E}$ becomes trivial, $H_{\mathcal{E}|U}$ is given by the morphism defined by the ring map $R[X_1, ..., X_n] \to R[T, X_1, ..., X_n]$ given by $X_j \mapsto TX_j$. Then for any map of vector bundles $f : \mathcal{E} \to \mathcal{F}$, we have $H_{\mathcal{E}} \circ (f \times \mathrm{id}_{\mathbb{A}^1}) = f \circ H_{\mathcal{F}}$ and for a morphism $g : Y \to X$ of schemes, we have $g \circ H_{\mathcal{E}} = H_{g^*\mathcal{E}}$. It follows that $H_{\mathcal{V}}$ is equivariant, so it gives the desired elementary $\mathbb{A}^1$-homotopy □

We say that a equivariant motivic space X is *sectionwise countable* if for every G-scheme U, the simplicial set $X(U)$ is a countable simplicial set.

Lemma 2.4.5. *Assume that G**Sm/S** is countable. There is a fibrant replacement functor R for $\mathcal{M}^G_\bullet(S)$ with the flasque model structure such that if $\mathcal{X} \in \mathcal{M}^G_\bullet(S)$ is sectionwise countable, then so is $R\mathcal{X}$.*

Proof. This is true by the same argument as in the proof of [NS11, Prop. 6]. Let $I_{\mathbf{SSet}} = \delta\Delta^n \to \Delta^n$ be the generating cofibrations of **SSet**, J_{fl} the generating trivial cofibrations for the global flasque model structure,

$$\Lambda(\mathcal{S}) = \{f \Box i : f \in \mathcal{S}, i \in I_{\mathbf{SSet}}\}$$

and $J = \Lambda(\mathcal{S}) \bigcup J_{fl}$. We claim that J satisfies the hypothesis of Prop. 1.2.3:

1. J is countable as $G\mathbf{Sm/S}$ is.
2. J permits the small object argument.
3. Let X be sectionwise countable, and F be a domain in J. If

$$F = \cup\mathcal{U}_+ \wedge \Delta^n_+ \coprod_{\cup\mathcal{U}_+ \wedge \Lambda^n_{k+}} X_+ \wedge \Lambda^n_{k+},$$

   then $\mathcal{M}^G_\bullet(S)(F, X)$ is countable as X is sectionwise countable. If $F \in \Lambda(\mathcal{S})$, then it is a finite pushout of tensors between representables and finite simplicial sets, so $\mathcal{M}^G_\bullet(S)(F, X)$ is countable.
4. If F is a codomainin J, it is again a finite pushout of tensors between representables and finite simplicial sets, so $\mathcal{M}^G_\bullet(S)(F, X)$ is countable.

Using the functorial factorization of Prop. 1.2.3, we obtain maps $\mathcal{X} \xrightarrow{\iota} R\mathcal{X} \xrightarrow{\pi} *$, with $R\mathcal{X}$ sectionwise countable. Since J_{fl} is a set of generating trivial cofibrations in $\mathcal{M}^G_\bullet(S)$ with the global structure, and every map in $\mathcal{S}$ are cofibrations with cofibrant domain in the global strucure, it follows that maps in $\Lambda(\mathcal{S})$ are trivial cofibrations in the equivariant motivic strucure. Hence, the relative J-cell complex ι obtained by the small object argument is a trivial cofibration. As π by construction has the right lifting property with respect to J_{fl}, it is fibrant in the global flasque structure. As it has the right lifting property with respect to J, by [Hir03, Prop. 4.2.4], it is $\mathcal{S}$-local, and hence fibrant in the equivariant motivic structure. □

Definition 2.4.6. The subcategory $\mathcal{M}^G(S)^{ft}$ of equivariant motivic spaces of *finite type* is the smallest full subcategory of $\mathcal{M}^G(S)$ such that

1. Every G-scheme is an object of $\mathcal{M}^G(S)^{ft}$.
2. If X, Y and Z are of finite type, the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

   is a pushout and i is a monomorphism, then $W \in \mathcal{M}^G(S)^{ft}$.

The category $\mathcal{M}^G_\bullet(S)$ is the full subcategory of pointed, equivariant motivic spaces $* \to X$ such that $X \in \mathcal{M}^G(S)^{ft}$

The following proposition is an equivariant analogue to [NS11, Thm 9].

Proposition 2.4.7. *Suppose $G\mathbf{Sm/S}$ is countable, let $X \in \mathcal{M}^G_\bullet(S)^{ft}$, and let $Y \in \mathcal{M}^G_\bullet(S)$ be sectionwise countable. Then, for all $n \geq 0$,*

$$\pi_n \underline{\mathcal{M}^G_\bullet(S)}(X, RY)$$

is countable.

Proof. By the same argument as in the proof of [NS11, Thm. 9], we can reduce to the case where $X \in G\mathbf{Sm/S}$ is a representable space. Then, using the fibrant replacement functor R given by 2.4.5, we get that

$$\underline{\mathcal{M}^G_\bullet(S)}(X, RY) \subseteq \underline{\mathcal{M}^G(S)}(X, RY) \simeq RY(X),$$

which is a countable Kan complex. □

2.4.2 The stable equivariant model category

Assume $S = k$ is a field such that the characteristic of k does not divide the order of the finite group G. To construct a stable homotopy theory for schemes, we follow the constructions in Sec. 1.4. Hence, we need an appropriate endofunctor.

Let V be a vector space over k. We write $\mathbb{A}(V) = \mathrm{Spec}(\mathrm{Sym}(V^\vee)$ for the its associated affine scheme. If $V \in G\mathbf{Sm/S}$ is a representation of G, define the *representation sphere* of V to be

$$S^V = \mathbb{A}(V)/(\mathbb{A}(V) - 0).$$

In the special case of $k[G]$, the regular representation of G, we use the notation $\mathbb{T}_G$, which we call the *regular representation sphere*. The category of *equivariant motivic spectra* is the category $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}^G_\bullet(k), \mathbb{T}_G)$. The homotopy category of $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}^G_\bullet(k))$ is called the *stable equivariant motivic homotopy category*, we denote it by $\mathrm{SH}^G(k)$.

Remark. As noted in the discussion on spectra for general model categories, this will not give us a model of spectra in which the smash product makes $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}^G_\bullet(k), \mathbb{T}_G)$ into a symmetric monoidal category. As this is required for some applications, other authors [HKO10; HVØ15] use symmetric spectra in the sense of either Jardine or Hovey. For our purposes, it is simpler to work with spectra. By the next proposition, we get the same stable homotopy category.

Proposition 2.4.8. *The cyclic permutation on $\mathbb{T}_G \wedge \mathbb{T}_G \wedge \mathbb{T}_G$ is homotopic to the identity.*

Proof. We first prove that there is a homotopy

$$\mathbb{A}^1 \times \mathbb{A}(G)^{\wedge 3} \to \mathbb{A}(G)^{\wedge 3}.$$

From the cyclical permutation to the identity. Note that the cyclical permutation is the action on $\mathbb{A}(G)^{\wedge 3}$ by the element of GL_3 represented by the matrix

$$c_{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

By multiplying with elementary matrices, we obtain an algebraic path $\omega : \mathbb{A}^1 \to GL_3$ from the identity matrix to $c_{1,2}$. Composing this path with the action of GL_3 on $\mathbb{A}(G)^{\wedge 3}$ gives a morphism of schemes

$$\mathbb{A}^1 \times \mathbb{A}(G)^3 \xrightarrow{\omega\times\mathrm{id}} GL_3 \times \mathbb{A}(G)^3 \to \mathbb{A}(G)^3$$

from the cyclic permutation to the identity. Passing to the quotient schemewise in **SSet** gives the desired homotopy

$$\mathbb{A}^1 \times \mathbb{T}_G^3 \to \mathbb{T}_G^3.$$

□

The next proposition will allow us to use the machinery of triangulated categories in Chapter 3.

Proposition 2.4.9. *There is a motivic weak equivalence $\mathbb{T}_G \simeq S^1 \wedge (\mathbb{A}(G) - 0)$ in $\mathcal{M}^G(k)_\bullet$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\delta\Delta^1 \wedge (\mathbb{A}(G) - 0) \arrow[r, hook] \arrow[d, hook] & \mathbb{A}(G) \arrow[r, "\simeq"] \arrow[d] & * \arrow[d] \\
\Delta^1 \wedge (\mathbb{A}(G) - 0) \arrow[r] \arrow[d, "\simeq"] & P \arrow[r] \arrow[d] & S^1 \wedge (\mathbb{A}(G) - 0) \\
* \arrow[r] & \mathbb{T}_G &
\end{tikzcd}$$

where every square is a pushout. The two maps decorated with tilde are weak equivalences. As both morphisms into the pushout P are cofibrations, it follows from properness that there is a zig-zag

$$\mathbb{T}_G \xleftarrow{\simeq} P \xrightarrow{\simeq} S^1 \wedge (\mathbb{A}(G) - 0)$$

of weak equivalences. □

Lemma 2.4.10. *The prolongation of $- \wedge S^1$ is a Quillen equivalence in $Sp^{\mathbb{N}}(\mathcal{M}^G(k)_\bullet, \mathbb{T}_G)$.*

Proof. By Thm. 1.4.5 and Prop. 2.4.9 it follows that smashing with $S^1 \wedge \mathbb{A}(G)$ is a Quillen equivalence, and that this is compatible with the tensored structure on $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}^G_\bullet(k), \mathbb{T}_G)$. By [Hov99, Prop. 1.3.13], this is equivalent with the derived functor defined by the action inducing a self-equivalence on $\mathrm{SH}^G(k)$. Hence, it suffices to show that if $\mathcal{C}$ is a symmetric monoidal category, $\mathcal{D}$ is tensored over $\mathcal{C}$ and there are objects $a, b \in \mathcal{C}$ such that the functor $F_{a\otimes b} = - \otimes (a \otimes b)$ is a self-equivalence on $\mathcal{D}$, then the functor F_a is also a self-equivalence.

Let $\mathcal{C}$ be symmetric monoidal and $a, b \in \mathcal{C}$ such that the functor $F_{a\otimes b} = -(a \otimes b)$ is an equivalence of categories. Let $G_{a\otimes b}$ denote its inverse. We claim that the functor $G_a := F_b \circ G_{a\otimes b}$ is an inverse for F_a. From the symmetric monoidal structure on $\mathcal{C}$, there are natural isomorphisms

$$F_a \circ F_b \cong F_{a\otimes b} \cong F_b \circ F_a.$$

As we have an isomorphism $F_{a\otimes b} \circ G_{a\otimes b} \cong \mathrm{id}$, we have an isomorphism

$$F_a \circ G_a = F_a \circ (F_b \circ G_{a\otimes b}) \cong F_{a\otimes b} \circ G_{a\otimes b} \cong \mathrm{id}$$

Furthermore, applying $G_{a\otimes b}$ on both sides of the natural isomorphism

$$F_{a\otimes b} \circ F_b \cong F_b \circ F_{a\otimes b}$$

gives a diagram of natural isomorphisms

$$\begin{tikzcd}
G_{a\otimes b} \circ F_{a\otimes b} \circ F_b \circ G_{a\otimes b} \arrow[r, "\cong"] \arrow[d, "\cong"] & G_{a\otimes b} \circ F_b \circ F_{a\otimes b} \circ G_{a\otimes b} \arrow[d, "\cong"] \\
\mathrm{id} \circ F_b \circ G_{a\otimes b} \arrow[r, "\cong"] \arrow[d, "\cong"] & G_{a\otimes b} \circ F_b \circ \mathrm{id} \arrow[d, "\cong"] \\
F_b \circ G_{a\otimes b} \arrow[r, "\cong"] & G_{a\otimes b} \circ F_b
\end{tikzcd}$$

Using the lower natural isomorphism, we deduce

$$\begin{aligned}
G_a \circ F_a &= F_b \circ G_{a\otimes b} \circ F_a \\
&\cong G_{a\otimes b} \circ F_b \circ F_a \\
&\cong \mathrm{id}
\end{aligned}$$

□

Lemma 2.4.5 can be extended to a stable analogue. The proof is similar to Naumann and Spitzweck's, but spell out some of the details.

Lemma 2.4.11. *There is a fibrant replacement functor for $Sp^{\mathbb{N}}(\mathcal{M}^G(k)_\bullet, \mathbb{T}_G)$ with the strict model structure such that if E is a $\mathbb{T}_G$-spectrum which is levelwise sectionwise countable, then its fibrant replacement is also levelwise sectionwise countable.*

Proof. Let R denote the fibrant replacement functor for the unstable motivic model structure provided by lemma 2.4.5. We construct a fibrant spectrum E' inductively by setting $E'_0 = (RE)_0$, and for $n > 0$

$$E'_{n+1} = R(E'_n \wedge \mathbb{T}_G \coprod_{E_n \wedge \mathbb{T}_G} E_{n+1}).$$

In other words, for $n > 0$, we have a pushout square in $\mathcal{M}^G(k)_\bullet$

$$
\begin{tikzcd}
E_n \wedge \mathbb{T}_G \arrow[r] \arrow[d] & E_{n+1} \arrow[d] \arrow[ddr, bend left] & \\
E'_n \wedge \mathbb{T}_G \arrow[r] \arrow[drr, bend right] & P_{n+1} \arrow[dr, hook, "\simeq"] & \\
& & E'_{n+1}
\end{tikzcd}
$$

where P_{n+1} denotes the pushout. By assumption, P_{n+1} is the pushout of two sectionwise countable equivariant motivic spaces. Hence, it is sectionwise countable, and thus E'_{n+1} is sectionwise countable by Lemma 2.4.5. The maps $E'_n \wedge \mathbb{T}_G \to E'_{n+1}$ makes E' into a spectrum, and by construction it is levelwise fibrant. Furthermore, we get a map $E \to E'$ as the structure maps of E' are compatible with maps $E_n \to E'_n$. Finally, we show that this is a strict trivial cofibration. Let $p : X \to Y$ be a strict fibration of spectra. For every n, we must show that there is a lift h_n in the diagram

$$
\begin{tikzcd}
E_n \arrow[r, "f_n"] \arrow[d] & X_n \arrow[d, "p_n"] \\
E'_n \arrow[r, "g_n"] \arrow[ur, dashed, "h_n"] & Y_n
\end{tikzcd}
$$

compatible with the structure maps. To find a lift h_0 poses no problem, as $E_0 \to E'_0$ is a trivial cofibration and p_0 is a fibration. Assume inductively that we have constructed a lift h_n. By the pushout property, a lift in the diagram

$$
\begin{tikzcd}
E'_n \wedge \mathbb{T}_G \coprod_{E_n \wedge \mathbb{T}_G} E_{n+1} \arrow[r, "(\sigma \circ \Sigma_g(h_n), f_{n+1})"] \arrow[d] & X_n \arrow[d, "p_n"] \\
E'_{n+1} \arrow[r, "g_n"] & Y_n
\end{tikzcd}
$$

constitutes a lift h_{n+1} compatible with the structure maps. By construction, the left vertical map is a trivial cofibration, so there is such a lift. $\square$

Chapter 3

Triangulated categories and Brown representability

3.1 Triangulated categories

Definition 3.1.1 (Triangle)**.** Let $\mathcal{T}$ be an additive category and let $\Sigma : \mathcal{T} \to \mathcal{T}$ be a functor. A *candidate triangle* in $\mathcal{T}$ consists of objects X, Y and Z in $\mathcal{T}$ and maps

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that $v \circ u = 0$ and $w \circ v = 0$.

A map of candidate triangles is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

where the rows are candidate triangles.

Definition 3.1.2. A *triangulated category* is an additive category $\mathcal{T}$ with an invertible functor $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of candidate triangles, called *distinguished triangles*, such that the following axioms hold:

TR0 A candidate triangle isomorphic to a distinguished triangle is a distinguished triangle. For any X in $\mathcal{T}$, the candidate triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma X$ is distinguished.

TR1 For any map $f : X \to Y$ in $\mathcal{T}$, there is a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

TR2 If either

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

or

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is a distinguished triangle, so is the other.

TR3 For every commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

where the rows are distinguished triangles, there is a map $h : Z \to Z'$, not necessarily unique, extending the diagram into a map of distinguished triangles.

TR4 Suppose $f : X \to Y$ and $g : Y \to Y'$ are maps in $\mathcal{T}$ and we are given triangles

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

$$X \xrightarrow{gf} Y' \longrightarrow Z' \longrightarrow \Sigma X$$

$$Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow \Sigma Y$$

Then the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X \arrow[r, "gf"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
0 \arrow[r] \arrow[d] & Y'' \arrow[r, "\mathrm{id}"] \arrow[d] & Y'' \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

is commutative, and every row and column is a distinguished triangle.

Given a cardinal α, a triangulated category $\mathcal{T}$ is said to satisfy **TR5(α)** if, for every set Λ of cardinality less than α and any family $\{X_\lambda : \lambda \in \Lambda\}$ the coproduct $\coprod_\lambda X_\lambda$ exists in $\mathcal{T}$. If $\mathcal{T}$ satisfies **TR5(α)** for every cardinal α, then $\mathcal{T}$ is said to satisfy **TR5**. In the following, we will assume all triangulated categories to satisfy **TR5**.

3.1.1 Cofiber sequences and trianglated categories

Given a map of based topological spaces $f : X \to Y$, there is a homotopy cofiber C_f given by $C_f = Y \coprod_f CX$, where CX is the cone of X. There is a natural map $Y \to C_f$, and the quotient of this map gives a map $C_f \to C_f/Y \cong \Sigma X$, where Σ is the functor given by smashing with S^1. This gives us a sequence

$$X \xrightarrow{f} Y \to C_f \to \Sigma X,$$

which descends to a candidate triangle in $\operatorname{Ho}\mathbf{Top}_*$. What is more, for any topological spaces X, the object ΣX is a cogroup object in the homotopy category (i.e., the functor $[\Sigma X, -]$ factors through **Grp**), and $\Sigma^2 X$ is an abelian cogroup object. For any based space Z, there is a long exact sequence [May99, Chapter 8] of based sets, groups and abelian groups

$$\cdots \to [\Sigma^2 X, Z] \to [\Sigma C_f, Z] \to [\Sigma Y, X] \to [\Sigma X, Z] \to [C_f, Z] \to [Y, X] \to [X, Z]$$

which extends indefinitely.

In Chapter 6 of [Hov99], the notion of cofiber sequences is extended to an arbitrary pointed simplicial category. Suppose $\mathcal{C}$ is a pointed simplicial model category, and let Σ denote smashing with S^1. As in $\mathbf{Top}_*$, any object ΣX is a homotopy cogroup object. If $f : X \to Y$ is a map in $\mathcal{C}$, define its *cofiber* C_f to be the coequalizer of f with the zero map. Hovey then constructs a natural group action

$$[C_f, W] \times [\Sigma X, W] \to [C_f, W]$$

for every cofibration $f : X \to Y$ between cofibrant objects X and Y and fibrant objects W. By naturality, the action is induced by a coaction, that is, a map $C_f \coprod \Sigma X \to \Sigma X$. Composing with the map $* \coprod \mathrm{id}_{\Sigma X}$ gives a sequence of maps

$$X \xrightarrow{f} Y \to C_f \to \Sigma X. \tag{3.1}$$

A *cofiber sequence* is any candidate triangle

$$X \to Y \to Z \to \Sigma X$$

in $\operatorname{Ho}\mathcal{C}$ isomorphic as a candidate triangle to a sequence on the form (3.1).

Theorem 3.1.3. *Suppose $\mathcal{M}$ is a pointed, simplicial model category such that Σ is a Quillen equivalence. Then $\operatorname{Ho}\mathcal{M}$ is a triangulated category with distinguished triangles given by the cofiber sequences.*

Proof. This follows from Prop. 7.1.6 in [Hov99]. ☐

Corollary 3.1.4. *$\mathrm{SH}^G(k)$ is a triangulated category.*

Proof. This follows from Lemma 2.4.10. ☐

3.2 Brown Representability in Triangulated Categories

If $\mathcal{T}$ is a triangulated category, then a functor $H : \mathcal{T}^{op} \to \mathbf{Ab}$ is said to be *exact* if it takes distinguished triangles to exact sequences, meaning that if

$$X \to Y \to Z \to \Sigma X$$

is a distinguished triangle, then

$$H(Z) \to H(Y) \to H(X)$$

is exact.

A *cohomological functor* is an exact functor $H : \mathcal{T}^{op} \to \mathbf{Ab}$ which takes coproducts to products, i.e. there is an isomorphism

$$H(\coprod_\alpha X_\alpha) \to \prod_\alpha H(X_\alpha)$$

for all coproducts $\coprod_\alpha X_\alpha$ in $\mathcal{T}$.

Remark. Some authors define cohomological functors to be what we call exact. We are following the terminology in [HPS97].

Example 3.2.1 (cf. [Nee01, Prop. 1.1.10])**.** Suppose $\mathcal{T}$ is a triangulated category and U an object of $\mathcal{T}$. Then $\mathrm{Hom}(-, U)$ is a cohomological functor. Indeed, if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is a distinguished triangle, we must have that the composition $\mathrm{Hom}(v, U) \circ \mathrm{Hom}(u, U)$ is 0. Suppose $f \in \mathrm{Hom}(Y, U)$ maps to 0 in $\mathrm{Hom}(X, U)$ under $\mathrm{Hom}(u, U)$. This means that the composition $X \xrightarrow{u} Y \xrightarrow{f} U$ is 0. Applying Σ, we get the commutative diagram

$$\begin{tikzcd}
\Sigma X \arrow[r, "-\Sigma u"] \arrow[d, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma v"] \arrow[d, "\Sigma f"] & \Sigma Z \arrow[r, "\Sigma w"] \arrow[d, dashed, "\Sigma h"] & \Sigma^2 X \arrow[d, "\Sigma f"] \\
0 \arrow[r] & \Sigma U \arrow[r, "\Sigma \mathrm{id}"] & \Sigma U \arrow[r] & \Sigma 0
\end{tikzcd}$$

The upper row is a distinguished triangle by **TR2**, while the lower is by **TR2** and **TR0**. By **TR3**, we get a map $\Sigma h : \Sigma Z \to \Sigma U$ making the diagram commute. Applying the self-equivalence Σ^{-1} to the square

$$\begin{tikzcd}
\Sigma Y \arrow[r, "\Sigma v"] \arrow[d, "\Sigma f"] & \Sigma Z \arrow[d, "\Sigma h"] \\
\Sigma U \arrow[r, "\Sigma \mathrm{id}"] & \Sigma U
\end{tikzcd}$$

gives a map $h : Z \to U$ which is mapped to f under $\mathrm{Hom}(v, U)$, so

$$H(Z) \xrightarrow{\mathrm{Hom}(v,U)} H(Y) \xrightarrow{\mathrm{Hom}(u,U)} H(X)$$

is exact.

One form of Brown representability is the question of whether every cohomological functor is representable (up to natural isomorphism). In order to make this tractable, we impose some cardinality conditions on our category.

An object X of a triangulated category $\mathcal{T}$ is said to be *compact* if, for every cardinal Λ and coproduct of objects $\coprod_{\lambda \in \Lambda} X_\lambda$ in $\mathcal{T}$ there is an isomorphism

$$\mathcal{T}(X, \coprod_{\lambda \in \Lambda} X_\lambda) \cong \coprod_{\lambda \in \Lambda} \mathcal{T}(X, X_\lambda).$$

Note that if X is compact, so is ΣX. A triangulated category is said to *compactly generated* if there exists a set C of compact objects of $\mathcal{T}$ such that

$$\mathcal{T}(C, X) = 0 \implies X = 0,$$

that is, if X is an object of $\mathcal{T}$ and $\mathcal{T}(T, X) = 0$ for all $T \in C$, then X must be the zero obect. A set C of objects in a compactly generated category is called a *generating set* if it is closed under Σ and

$$\mathcal{T}(C, X) = 0 \implies X = 0.$$

Remark. Suppose C is a set of compact objects such that

$$\mathcal{T}(C, X) = 0 \implies X = 0.$$

Then the set

$$\Sigma^* C = \{\Sigma^n T : T \in C, n \in \mathbb{Z}\}$$

is a generating set for $\mathcal{T}$.

Example 3.2.2. A spectra of CW-complexes X is stably equivalent to the trivial spectrum if all stable homotopy groups of X are trivial, in other words if

$$\pi_n(X) = [\Sigma^n \mathbb{S}, X] = 0$$

for all n. Thus, the stable homotopy category of CW-complexes is compactly generated, and the set $\cup_{n\in\mathbb{Z}} \Sigma^n \mathbb{S}$ is a generating set.

The previous example also follows from the following proposition, which gives way of producing generating sets whenever $\mathcal{T}$ is on the form in Theorem 3.1.3.

Proposition 3.2.3 ([Hov99, Thm. 7.3.1])**.** *Suppose $\mathcal{M}$ is a simplicial, pointed, cofibrantly generated model category such that $\Sigma : \mathcal{M} \to \mathcal{M}$ is a Quillen equivalence. Let I be the generating cofibrations of $\mathcal{M}$ and $\mathcal{G}$ the cofibers of I. Then $\Sigma^*\mathcal{G}$ is a generating set for* $\operatorname{Ho}\mathcal{M}$.

Corollary 3.2.4. $\mathrm{SH}^G(k)$ *is compactly generated. A compact generating set is given by taking all suspensions and desuspensions of the set*

$$\mathcal{G} = \left\{F_n^{\mathbb{T}_G}(X_+ \wedge S^1) : X \in G\mathbf{Sm}/k, n \in \mathcal{X}\right\}.$$

Proof. Recall that

$$I = \left\{(X_+ \wedge \delta\Delta^1 \to X_+ \wedge \Delta^1) : n \in \mathbb{Z}, X \in G\mathbf{Sm}/k\right\}$$

is a set of generating cofibrations for $\mathcal{M}^G(k)_*$, and hence

$$I_{\mathbb{T}_G} = \cup_{n\in\mathbb{Z}} F_n^{\mathbb{T}_G} I$$

is a set of generating cofibrations for $\mathrm{Sp}^{\mathbb{N}}(\mathcal{M}^G(k), - \wedge \mathbb{T}_G)$. To compute the cofiber of a map in $I_{\mathbb{T}_G}$, it suffices to compute the cofiber of the map

$$X_+ \wedge \mathbb{T}_G \wedge \delta\Delta^1_+ \to X_+ \wedge \mathbb{T}_G \wedge \Delta^1.$$

As this is the smash product with the identity and $\delta\Delta^1_+ \to \Delta^1_+$, its cofiber will be the smash product of $X_+ \wedge \mathbb{T}_G$ with the cofiber of $\delta\Delta^1_+ \to \Delta^1_+$, which is S^1. Hence, the set of cofibers of $I_{\mathbb{T}_G}$ is

$$\mathcal{G} = \left\{F_n^{\mathbb{T}_G}(X_+ \wedge S^1) : X \in G\mathbf{Sm}/k, n \in \mathcal{X}\right\}.$$

By Lemma 2.2 in [Jar00], finite simplicial sets and representable motivic spaces are compact, and the smash product of compact objects are compact, hence the objects of $\Sigma^*\mathcal{G}$ are compact. $\square$

One version of Brown representability is the statement that every cohomological functor is representable.

Theorem 3.2.5 ([Nee96, Thm. 3.1])**.** *If $\mathcal{T}$ is a compactly generated, every cohomological functor on $\mathcal{T}$ is representable.*

As we have seen, $\mathrm{SH}^G(k)$ satisfies this form of Brown representability.

If $\mathcal{T}$ is a triangulated category, a subcategory $\mathcal{S}$ of $\mathcal{T}$ is said to be *triangulated* if it is full and closed under suspension, isomorphisms and distinguished triangles, the latter meaning that if two of X, Y and Z are in $\mathcal{S}$ and

$$X \to Y \to Z \to \Sigma X$$

is a distinguished triangle in $\mathcal{T}$, then so is the last. The subcategory $\mathcal{T}^c$ of compact objects is a triangulated category. We say that $\mathcal{T}$ satisfies *Brown representability* if the following conditions are satisfied

- For every cohomological functor $H : (\mathcal{T}^c)^{op} \to \mathbf{Ab}$ there is an object $X \in \mathcal{T}$ and a natural isomorphism $H \xrightarrow{\cong} \mathcal{T}(-, X)_{|\mathcal{T}^c}$.
- Every natural transformation of functors $F : \mathcal{T}(-, X)_{|\mathcal{T}^c} \to \mathcal{T}(-, X)_{|\mathcal{T}^c}$ is induced by a map $X \to Y$ in $\mathcal{T}$.

In [Nee97, Thm 5.1, Prop. 4.11], Neeman shows that a compactly generated triangulated category satisfes Brown representability if it is equivalent to a countable category.

A subcategory $\mathcal{S}$ of a triangulated category is said to be *thick* if it is triangulated and closed under retracts.

Lemma 3.2.6 ([Nee92, Lemma 2.2])**.** *If $\mathcal{G}$ is a set of compact generators for $\mathcal{T}$, then the thick subcategory generated by $\mathcal{G}$ is equal to $\mathcal{T}^c$.*

Lemma 3.2.7. *If $\mathcal{S}$ is a countable subcategory of $\mathcal{T}$, then the thick subcategory generated by $\mathcal{S}$ is countable.*

Proof. Assume without loss of generality that $\mathcal{S}$ is closed under suspensions and desuspensions. We will inductively construct a filtration $\{\mathcal{S}_k : k \geq 0\}$ of the thick subcategory generated by $\mathcal{S}$. Let $\mathcal{S}_0 = \mathcal{S}$. Assume that $\mathcal{S}_k$ has been constructed and is a countable full subcategory of $\mathcal{T}$, closed under suspensions and desuspensions. If X is an object of $\mathcal{S}_k$, then a retract of X is an object $Y \in \mathcal{T}$ and maps $Y \xrightarrow{i} X \xrightarrow{p} Y$ such that $p \circ i = \mathrm{id}_Y$, so it correspond to the idempotent $e = i \circ p \in \mathcal{T}(X, X)$. By assumption, there are only a countable number of retracts of objects of $\mathcal{S}_k$ up to isomorphism. Pick one for each isomorphism class. Similarily, there are only a countable number of maps $f : X \to Y$ between objects in $\mathcal{S}$, so there are only a countable number of cofibers of such maps, up to isomorphism . Pick one for each isomorphism class. Let $\mathcal{S}_{k+1}$ be the full subcategory of $\mathcal{T}$ whose objects are either one of these choices or in $\mathcal{S}_k$.

It follows that $\mathcal{S}_{k+1}$ has countably many objects. To see that it is a countable subcategory, we therefore have to show that the sets of maps into or out of either a retract of an object or cofiber of a map in $\mathcal{S}_k$ is countable. Suppose

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "p"] \arrow[d, dashed, "g"] & Y \arrow[d, "f"] \\ Y' \arrow[r, "i'"] & X' \arrow[r, "p'"] & Y' \end{tikzcd}$$

is a diagram where Y and Y' are retracts of X and X', respectively, X and X' are in $\mathcal{S}_k$ and the solid maps commute. Letting $g = i' \circ f \circ p$ makes the whole diagram commute, and hence, it follows that there is an injective map $\mathcal{T}(Y, Y') \to \mathcal{T}(X, X')$, so the former is countable as the latter is.

If $X \xrightarrow{f} Y \to Z \to \Sigma X$ is a cofiber sequence with $f \in \mathcal{S}_k$, and U is an object of $\mathcal{S}_k$, then since the functor $\mathcal{T}(-, U)$ is exact, it follows that there is an exact sequence

$$\mathcal{T}(\Sigma X, U) \to \mathcal{T}(Z, U) \to \mathcal{T}(Y, U).$$

By assumption, the two groups on the sides are countable, it follows that $\mathcal{T}(Z, U)$ is as well. The same argument applies to $\mathcal{T}(U, Z)$.

We claim that $\bigcup_{k \geq 0} \mathcal{S}_k$ is equivalent to the thick subcategory generated by $\mathcal{S}$. By construction, it is a thick subcategory. On the other hand, any thick subcategory containing $\mathcal{S}$ must also contain $\mathcal{S}_1$, and inductively, every $\mathcal{S}_k$, so the claim follows. □

Corollary 3.2.8. *If $\mathcal{G}$ is a countable set of compact generators for $\mathcal{T}$ such that the full subcategory whose objects are $\mathcal{G}$ is a countable category, then $\mathcal{T}$ satisfies Brown representability.*

Theorem 3.2.9. *If $G\mathbf{Sm}/k$ is countable, then $\mathrm{SH}^G(k)$ satisfies Brown representability.*

Proof. From the proof of Corollary 3.2.4, we know that the set

$$\mathcal{G} = \left\{ \Sigma^n F_m^{\mathbb{T}_G}(X_+) : X \in G\mathbf{Sm}/k_*, n, m \in \mathbb{Z} \right\}$$

is a set of compact generators for $\mathrm{SH}^G(k)$. As it is countable, it follows from Cor. 3.2.8 that $\mathrm{SH}^G(k)$ satisfies Brown representability if

$$\mathrm{SH}^G(k)(F_n^{\mathbb{T}_G}(X_+), \Sigma^p F_q^{\mathbb{T}_G}(Y_+))$$

is countable for $n, p, q \in \mathbb{Z}$ and $X, Y \in G\mathbf{Sm}/k$. Let $E = \Sigma^p F_q^{\mathbb{T}_G}(Y_+)$. By [Hov01, Cor. 4.10], there is an isomorphism

$$\mathrm{SH}^G(k)(F_n^{\mathbb{T}_G}(X_+), E) \cong \mathrm{colim}_m\, H_\bullet^G(X_+ \wedge \mathbb{T}_G^n, E_{n+m}).$$

Assuming n to be big enough, the motivic space $E_{n+m} = Y_+ \wedge \mathbb{T}_G^{p+q+m} \wedge S^{p+k+m}$ is sectionwise countable, and $X_+ \wedge \mathbb{T}_G^n$ is of finite type, hence by 2.4.7 and [Hir03, Prop.9.5.3], it follows that $H_\bullet^G(X_+ \wedge \mathbb{T}_G^n, Y_+ \wedge \mathbb{T}_G^{q+k+m} \wedge S^{p+k+m})$ is countable. □

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